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Kelvin–Helmholtz instability under horizontal rotation and magnetic fields

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The author's previous work on the Rayleigh–Taylor instability is extended to the Kelvin–Helmholtz instability, and the maximum growth rate of a perturbation and an estimate of its upper bound is obtained for an infinite fluid layer under horizontal rotation where the density, horizontal velocity (shear) and magnetic field are continuously stratified in the direction of gravity. Conclusions are drawn about the possibility of stability for some directions of propagation of the perturbation, even in the case of unstably stratified density. It is also shown that the new terms that appear owing to the interaction of the horizontal shear flow, horizontal rotation and stratified magnetic field increase the range of values that contribute to the estimate of the maximum growth rate compared with previous work. Furthermore, a generalization of the sufficient condition for stability under horizontal rotation alone obtained by Johnson is calculated in the presence of density stratification. A new method is also given to obtain a sufficient condition for stability when a magnetic field is present in addition to rotation and density stratification.

1. Introduction

The Kelvin–Helmholtz instability (KHI) due to shear flow in stratified fluids has attracted the attention of many researchers because of its determinant influence on the stability of planetary and stellar atmospheres and in practical applications. In particular, the effects of rotation and magnetic fields in KHI of ideal incompressible fluids have been investigated (Chandrasekhar 1961). The instability of a parallel shear flow in an inviscid homogeneous unbounded rotating fluid was investigated by Johnson (1963). By means of a converse of the Taylor–Proudman theorem, he showed that for an incompressible inviscid fluid the rotation vector cannot be in the direction of velocity stratification. He calculated the stability curves for the hyperbolic-tangent velocity profile. His results showed that three-dimensional perturbations are stabilized in the cyclonic case but destabilized in the anticyclonic case. More recently, Yanase *et al.* (1993), supposing a viscous fluid, calculated the stability of the hyperbolic-tangent (mixing) and Gaussian bell-shaped (wake) velocity profiles under rotation perpendicular to the velocity field but in the same shear layer. They found that cyclonic rotation and strong anticyclonic rotation tend to stabilize the three-dimensional motions but that weak anticyclonic rotation destabilizes these perturbations. In this sense, they corrected some results of Johnson (1963). Shear-flow instabilities have also been investigated in stratified fluids.

For example, Jones (1967) investigated the propagation of internal gravity waves in fluids with shear flow and rotation but including density stratification in the horizontal and vertical directions. This problem has also been investigated by Teitelbaum *et al.* (1987). They made a comparison between the results for over-reflection obtained from the Boussinesq and the hydrostatic approximations. Sumathi and Raghavachar (1993) investigated a similar system but with the goal of understanding the effects of three-dimensional perturbations on the stability of a general vertically stratified velocity field. The effect of a magnetic field on the stability of shear flows was investigated in the absence of density stratification by Chen and Morrison (1991). In their paper the shear flow was taken to be antisymmetric, with symmetric and antisymmetric stratification of the magnetic field. They showed that a stable Couette flow may be driven unstable by a symmetric magnetic field. For the hyperbolic-tangent profile they showed that a strong magnetic-field stratification may cause instability. The combined effect of density stratification, rotation and magnetic field was investigated by Rudraiah and Venkatachalappa (1972) and by El Mekki (1982). Rudraiah and Venkatachalappa (1972) considered rotation to be vertical and the uniform magnetic field to be horizontal in the shear flow layer. They found that there are seven singularities (singular levels) in the hydromagnetic flow, in comparison with three in hydrodynamic flow. They obtained asymptotic solutions of their wave equation near the critical levels, and showed that the Lorentz force increases wave absorption at the critical levels. El Mekki (1982), under the supposition of a horizontal stratified magnetic field in the x direction and vertical rotation, investigated the relation of zonal-magnetic shears and the amplitude of hydromagnetic planetary gravity waves. El Mekki found that wind shear destabilizes but that a magnetic field decreases the wave amplitude. An interesting experiment with mercury by Amaguchi *et al.* (1991) included, in the absence of density stratification, the ingredients of shear flow perpendicular with respect to parallel rotation and magnetic fields. When the basic motion is two-dimensional, they found that a vortex structure appears in a discontinuous velocity field owing to KHI.

In the absence of main flow or shear, the Rayleigh–Taylor instability (RTI) is left as the source of perturbation growth in the case of an unstably stratified medium. Recently, new advances in RTI for non-dissipative incompressible fluids have been made in which a continuously stratified fluid is subjected to a general rotation field (Dávalos-Orozco and Aguilar-Rosas 1989*a*) and, moreover, to a general stratified horizontal magnetic field (Dávalos-Orozco and Aguilar-Rosas 1989*a*; Dávalos-Orozco 1991). In those papers, the maximum growth rate of the instability was obtained along with an estimate of the upper bound of the growth rate. It was shown in Dávalos-Orozco and Aguilar-Rosas (1989*a*) that the maximum growth rate in the presence of a general rotation field depends on the wavenumber as well as on rotation, but that the upper bound on the growth rate is independent of these parameters. When a horizontal vertically stratified magnetic field is included, it was shown (Dávalos-Orozco and Aguilar-Rosas 1989*b*; Dávalos-Orozco 1991) that the maximum growth rate reduces to that of general rotation alone when the direction of propagation of the perturbation is perpendicular to the magnetic field. This is related to results previously obtained by Chandrasekhar (1961) for a two-layer system. Calculations of the RTI of a two-fluid system under horizontal rotation and magnetic field was calculated analytically and numerically by Dávalos-Orozco (1993).

In this paper, adapting the method used by Dávalos-Orozco and Aguilar Rosas (1989*a*, *b*), calculations are performed to obtain the maximum growth rate of the perturbation and an estimate of its upper bound for the Kelvin–Helmholtz instability of an infinite fluid layer under horizontal rotation where the density, horizontal velocity and horizontal magnetic field are continuously stratified in the direction of gravity. To the best of the author's knowledge, the problem with this system configuration has not previously been investigated. Moreover, a generalization of the sufficient condition for stability obtained by Johnson (1963) is calculated in the presence of density stratification. Besides, a new method is presented to obtain a sufficient condition for stability when a magnetic field is added to rotation and density stratification. This problem corresponds physically to the KHI of an equatorial section of a planetary magnetosphere or of a stellar atmosphere where shear, rotation and magnetic field are perpendicular to gravity.

In the next section, the equations of motion of the perturbed flow are obtained from the Euler equations in the Boussinesq approximation. These equations are recovered in an integral form, from which the maximum growth rate and its upper bound are obtained. In Sec. 3, a generalization of the sufficient condition for stability obtained by Johnson (1963) is calculated in the presence of density stratification, and a new method is presented to obtain a sufficient condition for stability when a magnetic field is present together with all the other effects. The conclusions are given in Sec. 4.

2. Equations of motion for the perturbation

The system is supposed to be a non-dissipative incompressible infinite magnetofluid layer with density stratified in the z direction parallel to gravity. Rotation, magnetic field and velocity are in a horizontal plane. The velocity and magnetic field are stratified in the vertical direction. The system whose stability is investigated is shown in Fig. 1. In what follows, the density, velocity and magnetic field stratifications are arbitrary. The equations of motion in the Boussinesq approximation are

$$\rho' \frac{D\mathbf{u}'}{Dt} + 2\rho' \boldsymbol{\Omega} \times \mathbf{u}' + \rho' \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\boldsymbol{\nabla} p' - \rho' g \hat{k} + \frac{\mu}{4\pi} (\boldsymbol{\nabla} \times \mathbf{H}') \times \mathbf{H}', \tag{1}$$

$$\frac{\partial \mathbf{H}'}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u}' \times \mathbf{H}'), \tag{2}$$

$$\frac{D\rho'}{Dt} = 0, \tag{3}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}' = 0, \tag{4}$$

$$\boldsymbol{\nabla} \cdot \mathbf{H}' = 0 \tag{5}$$

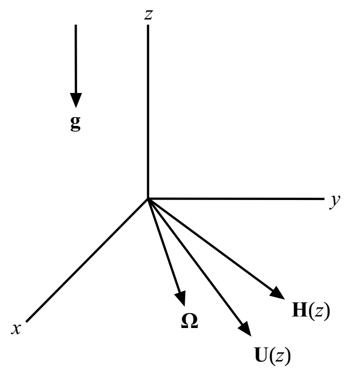

Figure 1. The system whose stability is investigated. The density is stratified in the z direction, parallel to gravity. Rotation is horizontal in an arbitrary direction, and the horizontal velocity and horizontal magnetic fields are also stratified in the z direction.

where D/Dt is the Lagrange operator, $\hat{\mathbf{k}} = (0, 0, 1)$, g is the acceleration due to gravity and the variables are defined as follows:

$$\left.\begin{aligned} \mathbf{u}' &= \mathbf{U}(z) + \mathbf{u}(x, y, z, t), \\ \mathbf{H}' &= \mathbf{H}_0(z) + \mathbf{h}(x, y, z, t), \\ p' &= p(z) + \delta p(x, y, z, t), \\ \rho' &= \rho(z) + \delta\rho(x, y, z, t). \end{aligned}\right\} \tag{6}$$

Here $\mathbf{u}'$, $\mathbf{H}'$, p' and ρ' are the velocity, the magnetic field, the pressure and the density respectively. From the above equations, the hydrostatic equilibrium is given by

$$2\rho\mathbf{\Omega}\times\mathbf{U} + \rho\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = -\boldsymbol{\nabla}p - \rho g\hat{\mathbf{k}} - \boldsymbol{\nabla}\left(\frac{\mu}{8\pi}|\mathbf{H}_0|^2\right), \tag{7}$$

where $\mathbf{\Omega} = (\Omega_x, \Omega_y, 0)$, $\mathbf{H}_0 = (H_{0x}(z),\ H_{0y}(z),\ 0)$ and $\mathbf{U} = (U(z),\ V(z),\ 0)$ are the imposed rotation and magnetic fields and the velocity field. The functions $\rho = \rho(z)$ and $p = p(z)$ are the unperturbed density and pressure.

When a perturbation is applied to the system, $\mathbf{u} = (u, v, w)$, $\mathbf{h} = (h_x, h_y, h_z)$, δp and $\delta\rho$ are the perturbations of velocity, magnetic field, pressure and density respectively, which satisfy the equations

$$\rho\left(\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + w\frac{\partial U}{\partial z}\right) + 2\rho\Omega_y w = -\frac{\partial\delta p}{\partial x} + \frac{\mu h_z}{4\pi}\frac{\partial H_{0x}}{\partial z} - \frac{\mu H_{0y}}{4\pi}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right), \tag{8}$$

$$\rho\left(\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + w\frac{\partial V}{\partial z}\right) - 2\rho\Omega_x w = -\frac{\partial\delta p}{\partial y} + \frac{\mu h_z}{4\pi}\frac{\partial H_{0y}}{\partial z} + \frac{\mu H_{0x}}{4\pi}\left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right), \tag{9}$$

$$\begin{aligned} \rho\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + V\frac{\partial w}{\partial y}\right) + 2\rho(\Omega_x v - \Omega_y u) &= -\frac{\partial\delta p}{\partial z} - g\delta\rho - \frac{\mu h_y}{4\pi}\frac{\partial H_{0y}}{\partial z} - \frac{\mu h_x}{4\pi}\frac{\partial H_{0x}}{\partial z} \\ &\quad + \frac{\mu H_{0y}}{4\pi}\left(\frac{\partial h_z}{\partial y} - \frac{\partial h_y}{\partial z}\right) - \frac{\mu H_{0x}}{4\pi}\left(\frac{\partial h_x}{\partial z} - \frac{\partial h_z}{\partial x}\right), \end{aligned} \tag{10}$$

$$\frac{\partial \delta\rho}{\partial t}+U\frac{\partial \delta\rho}{\partial x}+V\frac{\partial \delta\rho}{\partial y}+w\frac{\partial \rho}{\partial z}=0, \tag{11}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{12}$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0, \tag{13}$$

$$\frac{\partial \mathbf{h}}{\partial t}+U\frac{\partial \mathbf{h}}{\partial x}+V\frac{\partial \mathbf{h}}{\partial y}+w\frac{\partial \mathbf{H}}{\partial z}=\mathbf{H}_0\cdot\nabla\mathbf{u}+h_z\frac{\partial \mathbf{U}}{\partial z}. \tag{14}$$

A solution of (8)–(14) in normal modes is assumed, varying as

$$\exp[i(k_x x+k_y x)+nt], \tag{15}$$

where k_x and k_y are the x and y components of the wavenumber and $k = (k_x^2+k_y^2)^{1/2}$ is its magnitude. n is a complex number whose real and imaginary parts are the growth factor and the frequency respectively.

After substitution of all variables in the form (15) is made into (8)–(14), a combination of them leads to an equation for the vertical velocity component w alone, namely

$$\begin{aligned}&\left\{k^2\rho(n+i\mathbf{k}\cdot\mathbf{U})^2\left[1+\frac{\mu H^2}{4\pi\rho(n+i\mathbf{k}\cdot\mathbf{U})^2}\right]\right\}\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\\ &-\{2i\Omega^- D[\rho(n+i\mathbf{k}\cdot\mathbf{U})]+k^2gD\rho\}\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\\ &+\left[\frac{4\rho\Omega^{+2}}{1+\mu H^2k^2/4\pi\rho(n+i\mathbf{k}\cdot\mathbf{U})^2}-2\rho k^2(\Omega_x DV-\Omega_y DU)\right]\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\\ &-D\left\{\rho(n+i\mathbf{k}\cdot\mathbf{U})^2\left[1+\frac{\mu H^2}{4\pi\rho(n+i\mathbf{k}\cdot\mathbf{U})^2}\right]D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right\}=0,\end{aligned} \tag{16}$$

where $D = d/dz$, and the following definitions have been used:

$$H = k_x H_{0x}+k_y H_{0y}, \tag{17}$$

$$\Omega^- = k_y\Omega_x-k_x\Omega_y, \tag{18}$$

$$\Omega^+ = k_x\Omega_x+k_y\Omega_y. \tag{19}$$

Then, after multiplying (16) by the complex conjugate of

$$\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}},$$

that is, by

$$\frac{\bar{w}}{\bar{n}-i\mathbf{k}\cdot\mathbf{U}},$$

where the overbar means complex conjugate, we integrate over the range of z to arrive at

$$\int k^2\rho(n+i\mathbf{k}\cdot\mathbf{U})^2\left[1+\frac{\mu H^2k^2}{4\pi\rho(n+i\mathbf{k}\cdot\mathbf{U})^2}\right]\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz$$
$$-\int\{2i\Omega^- D[\rho(n+i\mathbf{k}\cdot\mathbf{U})]+k^2gD\rho\}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz$$
$$+\int\left[\frac{4\rho\Omega^{+2}}{1+\mu H^2k^2/4\pi\rho(n+i\mathbf{k}\cdot\mathbf{U})^2}-2\rho k^2(\Omega_x DV-\Omega_y DU)\right]\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz$$
$$+\int\left\{\rho(n+i\mathbf{k}\cdot\mathbf{U})^2\left[1+\frac{\mu H^2k^2}{4\pi\rho(n+i\mathbf{k}\cdot\mathbf{U})^2}\right]\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2\right\}dz=0. \quad (20)$$

Now, let us define the real and imaginary parts of n as

$$n=R+iI. \quad (21)$$

Then two equations for R and I may be obtained: one by subtracting (20) from its complex conjugate, and another by adding (20) to its complex conjugate. They are

$$RI\left(B^2+\frac{4}{k^2}E_1^2\right)+R\left(G-\frac{\Omega^-}{k^2}C+\frac{4}{k^2}E_2\right)=0, \quad (22)$$

$$(R^2-I^2)\left(B^2+\frac{4}{k^2}E_1^2\right)-2I\left(G-\frac{\Omega^-}{k^2}C+\frac{4}{k^2}E_2\right)$$
$$-J^2+\sigma^2-gC+2\frac{\Omega^-}{k^2}A+\frac{4}{k^2}E_3^2-\frac{4}{k^2}E_4^2=0. \quad (23)$$

Note that it is required that $R \neq 0$, in order for (22) be satisfied by I. Here the following definitions of the integrals have been used:

$$A=\int D\rho\mathbf{k}\cdot\mathbf{U}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz, \quad (24)$$

$$B^2=\int\rho\left(\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2+\frac{1}{k^2}\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2\right)dz\geqslant 0, \quad (25)$$

$$C=\int D\rho\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz, \quad (26)$$

$$E_1^2=\Omega^{+2}\int\rho^2\frac{\mu H^2}{4\pi}\frac{1}{|\rho(n+i\mathbf{k}\cdot\mathbf{U})^2+\mu H^2/4\pi|^2}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz$$
$$=\Omega^{+2}E_1'^2\geqslant 0, \quad (27)$$

$$E_2=\Omega^{+2}\int\rho^2\mathbf{k}\cdot\mathbf{U}\frac{\mu H^2}{4\pi}\frac{1}{|\rho(n+i\mathbf{k}\cdot\mathbf{U})^2+\mu H^2/4\pi|^2}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz, \quad (28)$$

$$E_3^2 = \Omega^{+2}\int \rho^3\,|n+i\mathbf{k}\cdot\mathbf{U}|^4 \frac{1}{|\rho(n+i\mathbf{k}\cdot\mathbf{U})^2+\mu H^2/4\pi|^2}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz$$

$$= \Omega^{+2}E_3'^2 \geqslant 0, \tag{29}$$

$$E_4^2 = \Omega^{+2}\int \rho^2(\mathbf{k}\cdot\mathbf{U})^2\frac{\mu H^2}{4\pi} \frac{1}{|\rho(n+i\mathbf{k}\cdot\mathbf{U})^2+\mu H^2/4\pi|^2}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz$$

$$= \Omega^{+2}E_4'^2 \geqslant 0, \tag{30}$$

$$G = \int \rho\mathbf{k}\cdot\mathbf{U}\left(\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 + \frac{1}{k^2}\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2\right)dz, \tag{31}$$

$$J^2 = \int \rho(\mathbf{k}\cdot\mathbf{U})^2\left(\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 + \frac{1}{k^2}\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2\right)dz \geqslant 0, \tag{32}$$

$$\sigma^2 = \int \frac{\mu H^2}{4\pi}\left(\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 + \frac{1}{k^2}\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2\right)dz \geqslant 0. \tag{33}$$

Note that some integrals are positive or zero, depending on the magnitude of one of their terms. This property will be important in determining the maximum growth rate.

Now, if the solution for I in (22) is substituted into (23), we have

$$\frac{\left(G-\dfrac{\Omega^-}{k^2}C+\dfrac{4}{k^2}E_2\right)^2}{B^2+\dfrac{4}{k^2}E_1^2}+\sigma_2^2+\frac{4}{k^2}E_3^2$$

$$= -R^2\left(B^2+\frac{4}{k^2}E_1^2\right)+J^2-\sigma_1^2+gC-\frac{2}{k^2}\Omega^-A+\frac{4}{k^2}E_4^2 \geqslant 0, \tag{34}$$

where the definition of σ^2 in (3) has been split into the following:

$$\sigma_1^2 = \int \frac{\mu H^2}{4\pi}\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz \geqslant 0, \tag{35}$$

$$\sigma_2^2 = \int \frac{\mu H^2}{4\pi k^2}\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2 dz \geqslant 0. \tag{36}$$

The left-hand side of (34) is positive or zero because all the terms in it have the same property. Consequently, the right-hand side must be positive or zero. Therefore the maximum growth rate of the perturbation is obtained from the right-hand side of (34) as follows:

$$R^2 \leqslant \left(J^2+\frac{4}{k^2}E_4^2+gC-\frac{2}{k^2}\Omega^-A-\sigma_1^2\right)\Big/\left(B^2+\frac{4}{k^2}E_1^2\right). \tag{37}$$

With this expression, an estimate of the upper bound of the growth rate may be calculated, namely

$$R^2 \leqslant M+\frac{M_1}{1+Q} < M+M_1, \tag{38}$$

where

$$M = \max(\mathbf{k}\cdot\mathbf{U})^2 \tag{39}$$

is used in the quotient

$$\left(J^2+\frac{4}{k^2}E_4^2\right)\Big/\left(B^2+\frac{4}{k^2}E_1^2\right)$$

and

$$M_1 = \max\left[\rho^{-1}\left(gD\rho-\frac{2}{k^2}\Omega^{-}D\rho\mathbf{k}\cdot\mathbf{U}-\frac{\mu H^2}{4\pi}\right)\right] \tag{40}$$

is used in the quotient

$$\left(gC-\frac{2}{k^2}\Omega^{-}A-\sigma_1^2\right)\Big/B_1^2.$$

The following definition has also been used:

$$Q = \frac{B_2^2}{B_1^2}+\frac{4}{k^2}\frac{E_1^2}{B_1^2} > 0, \tag{41}$$

where B^2 in (25) has been split into

$$B_1^2 = \int\rho\left|\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz \geqslant 0, \tag{42}$$

$$B_2^2 = \int\rho\frac{1}{k^2}\left|D\left(\frac{w}{n+i\mathbf{k}\cdot\mathbf{U}}\right)\right|^2 dz \geqslant 0. \tag{43}$$

Note that the magnitudes of the maxima M and M_1 depend not only on z and the wavenumber but also on the relative angles between $\mathbf{k}$, $\mathbf{U}$, $\boldsymbol{\Omega}$ amd $\mathbf{H}_0$.

2.1. Discussion of results

Here, a discussion of the maximum growth rate given in (37) is presented, trying to find sufficient conditions for stability. The discussion will be separated in different cases, starting with the more simple in order to understand the general result.

Before discussing the case with shear flow, we note that (37) reduces to that obtained for RTI in the absence of vertical rotation when the velocity is zero (Dávalos-Orozco and Aguilar-Rosas 1989*a*, *b*; Dávalos-Orozco 1991).

Case 1. It is supposed that

$$\Omega_x = \Omega_y = 0, \qquad H_{0x} = H_{0y} = 0, \qquad D\rho = 0,$$

There are no fields or density stratification. Then (37) reduces to

$$R^2 \leqslant \frac{J^2}{B^2} \leqslant M,$$

where

$$M = \max(\mathbf{k}\cdot\mathbf{U})^2.$$

In the absence of shear flow, the system is stable (nothing happens). Note that when the perturbation propagates perpendicularly to the shear flow, $\mathbf{k}\perp\mathbf{U}$, its effect does not appear and the growth rate is zero in that direction. The maximum growth rate will be attained at some value of z when the perturbation propagates in the same direction of the shear flow.

Case 2. The following conditions are given:

$$\Omega_x = \Omega_y = 0. \quad H_{0x} = H_{0y} = 0, \quad D\rho \neq 0.$$

There are no fields, and the fluid is stratified. Equation (37) becomes

$$R^2 \leqslant \frac{J^2 + gC}{B^2} \leqslant M + M_1,$$

where

$$M = \max(\mathbf{k}\cdot\mathbf{U})^2, \quad M_1 = \max[\rho^{-1}(gD\rho)].$$

Note that when the fluid is unstably stratified, the stratification slope $D\rho > 0$ also contributes to the growth rate. If the fluid is stably stratified, it may contribute to decrease the growth rate and even to stabilize the shear flow in all directions. When $\mathbf{k} \perp \mathbf{U}$, the stability is only governed by fluid density stratification.

Case 3. Here horizontal rotation is allowed:

$$\Omega_x \neq 0, \quad \Omega_y \neq 0, \quad H_{0x} = H_{0y} = 0, \quad D\rho \neq 0.$$

There is no magnetic field, and the fluid is stratified. The growth rate is

$$R^2 \leqslant \frac{J^2 + gC - \dfrac{2}{k^2}\Omega^- A}{B^2} \leqslant M + M_1,$$

where

$$M = \max(\mathbf{k}\cdot\mathbf{U})^2, \quad M_1 = \max\left[\rho^{-1}(gD\rho) - \frac{2}{k^2}\Omega^- D\rho\mathbf{k}\cdot\mathbf{U}\right].$$

The horizontal component of rotation may stabilize the flow in the angular region where $-(2/k^2)\,\Omega^- D\rho\mathbf{k}\cdot\mathbf{U} < 0$. This rotation term does not appear when $U = 0$, as shown for RTI (Dávalos-Orozco and Aguilar-Rosas 1989*a*). Moreover, it is zero if the perturbation propagates in the direction of rotation, $\mathbf{k} \parallel \boldsymbol{\Omega}$; that is, when $\Omega^- = 0$. Its strongest influence appears when the perturbation propagates perpendicularly to rotation. Note that this term also appears owing to density stratification, and it is not present in a homogeneous fluid. The sign differences among Ω^-, $D\rho$ and $\mathbf{k}\cdot\mathbf{U}$ are important to determine the stabilizing effect. These signs also depend on the relative directions among the vectors $\mathbf{k}$, $\mathbf{U}$ and $\boldsymbol{\Omega}$. Only when the sign product is negative may this term stabilize the system, no matter which density stratification is present (stable or unstable). For a given density gradient, the largest magnitude of this term occurs when $\mathbf{k} \perp \boldsymbol{\Omega}$ and $\mathbf{k} \parallel \mathbf{U}$.

Note that density plays a double role in the terms $gC - (2/k^2)\,\Omega^- A$. That is, when $D\rho < 0$, the term gC is negative and stabilizes, but if $-\Omega^-\mathbf{k}\cdot\mathbf{U} < 0$ at the same time, the term $(2/k^2)\,\Omega^- A$ destabilizes.

Case 4. A horizontal magnetic field is allowed:

$$\Omega_x = 0, \quad \Omega_y = 0, \quad H_{0x} \neq H_{0y} \neq 0, \quad D\rho \neq 0.$$

There is no rotation field, and the fluid is stratified. Now (37) is

$$R^2 \leqslant \frac{J^2 + gC - \sigma_1^2}{B^2} \leqslant M + M_1$$

where

$$M = \max(\mathbf{k}\cdot\mathbf{U})^2 \qquad M_1 = \max\left[\rho^{-1}(gD\rho) - \frac{\mu H^2}{4\pi}\right].$$

The stabilizing effect of the horizontal magnetic field in the absence of shear flow when $D\rho > 0$ has already been demonstrated (Chandrasekhar 1961; Dávalos-Orozco and Aguilar-Rosas 1989*b*; Dávalos-Orozco 1991). Here it works to oppose the effects of shear and gravity in the terms J^2 and gC respectively. However, it has no effect when the perturbation propagates perpendicularly to the magnetic field; that is, when $H = 0$ by definition, or $\mathbf{k}\perp\mathbf{H}_0$. If $\mathbf{U}\parallel\mathbf{H}_0$ and at the same time $\mathbf{k}\perp\mathbf{H}_0$ then the stability will only depend on $D\rho$. When $\mathbf{U}$ and $\mathbf{H}_0$ are not parallel, the effect of each will be null in different regions with respect to the wave vector.

Case 5. Both horizontal rotation and magnetic fields are included:

$$\Omega_x \neq 0, \quad \Omega_y \neq 0, \quad H_{0x} \neq H_{0y} \neq 0, \quad D\rho \neq 0$$

and the fluid is stratified. Now (37) is

$$R^2 \leqslant \frac{J^2 + \dfrac{4\Omega^{+2}}{k^2}E_4'^2 + gC - \dfrac{2}{k^2}\Omega^- A - \sigma_1^2}{B^2 + \dfrac{4\Omega^{+2}}{k^2}E_1'^2} \leqslant M + M_1$$

where

$$M = \max(\mathbf{k}\cdot\mathbf{U})^2, \qquad M_1 = \max\left[\rho^{-1}(gD\rho) - \frac{2}{k^2}\Omega^- D\rho\,\mathbf{k}\cdot\mathbf{U} - \frac{\mu H^2}{4\pi}\right].$$

Here the growth rate is presented in a different way to show explicitly the effect of rotation in the new terms that only appear when both rotation and magnetic field act simultaneously.

Note that the terms $(4\Omega^{+2}/k^2)E_4'^2$ and $-(2/k^2)\Omega^- A$ become zero at angles separated by 90° from each other; that is, the term $\Omega^+ = 0$ when $\mathbf{k}\perp\boldsymbol{\Omega}$, and the term $\Omega^- = 0$ when $\mathbf{k}\parallel\boldsymbol{\Omega}$, but the former always has a destabilizing effect. Now, the term σ_1^2 always stabilizes except when $H = 0$, that is if $\mathbf{k}\perp\mathbf{H}_0$.

Therefore the term $(4\Omega^{+2}/k^2)E_4'^2$ becomes zero in three directions defined by $\Omega^+ = 0$ when $\mathbf{k}\perp\boldsymbol{\Omega}$, $\mathbf{k}\cdot\mathbf{U} = 0$ when $\mathbf{k}\perp\mathbf{U}$, and $H = 0$ when $\mathbf{k}\perp\mathbf{H}_0$.

The terms $(4\Omega^{+2}/k^2)E_4'^2$ and $-(2/k^2)\Omega^- A$ show the dual role that horizontal rotation plays in the presence of a horizontal magnetic field and shear flow. It always destabilizes through the first term supported in some directions by the second, and it may stabilize through the second term in other directions.

3. Generalization of Johnson's sufficient condition for stability including density stratification, and a new method to obtain a sufficient condition for stability in the additional presence of a magnetic field

In his paper, Johnson (1963) obtained a sufficient condition for stability of a fluid under shear and horizontal rotation. The fluid was not stratified, and a magnetic field was not taken into account. In this section, the results of calculations are given showing that in the present problem a more general sufficient condition can be obtained for density stratification. However, we show that the generalization of Johnson's method to the case where a magnetic field is present cannot determine the role played by that field with regard to stability. Therefore a new method is developed to obtain a sufficient condition for stability in the presence of all the fields and density stratification.

The procedure followed by Johnson starts by replacing, in (16), the complex amplitude of the z component of velocity w by $f(n+i\mathbf{k}\cdot\mathbf{U})^{1/2}$. After some algebra, the resulting equation is

$$\begin{aligned} &D\left\{\left[\rho(n+i\mathbf{k}\cdot\mathbf{U})+\frac{\mu H^2/4\pi}{n+i\mathbf{k}\cdot\mathbf{U}}\right]Df\right\}-\tfrac{1}{2}\rho i\mathbf{k}\cdot D^2\mathbf{U}f \\ &+\frac{1}{4}\frac{\rho(\mathbf{k}\cdot D\mathbf{U})^2}{n+i\mathbf{k}\cdot\mathbf{U}}f \\ &-k^2\left[\rho(n+i\mathbf{k}\cdot\mathbf{U})+\frac{\mu H^2/4\pi}{n+i\mathbf{k}\cdot\mathbf{U}}\right]f-\frac{4\Omega^{+2}\rho^2(n+i\mathbf{k}\cdot\mathbf{U})f}{\rho(n+i\mathbf{k}\cdot\mathbf{U})^2+\mu H^2/4\pi} \\ &+2\rho\Omega^+D(k_x V-k_y U)\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}} \\ &+[gk^2+2i\Omega^-(n+i\mathbf{k}\cdot\mathbf{U})]D\rho\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}} \\ &-\frac{\mu}{4\pi}H^2 f\left[\frac{1}{2}\frac{i\mathbf{k}\cdot D^2\mathbf{U}}{(n+i\mathbf{k}\cdot\mathbf{U})^2}+\frac{3}{4}\frac{(\mathbf{k}\cdot D\mathbf{U})^2}{(n+i\mathbf{k}\cdot\mathbf{U})^3}\right]=0. \end{aligned} \tag{44}$$

Now, multiplication of the complex conjugate of f, that is, f^*, and integration of (44) over the full range of z lead us to

$$\begin{aligned} &\int\left[\rho(n+i\mathbf{k}\cdot\mathbf{U})+\frac{\mu H^2/4\pi}{n+i\mathbf{k}\cdot\mathbf{U}}\right](|Df|^2+k^2|f|^2)\,dz \\ &\quad+\int\tfrac{1}{2}\rho i\mathbf{k}\cdot D^2\mathbf{U}|f|^2\,dz \\ &\quad+\int\left[\frac{4\Omega^{+2}\rho^2(n+i\mathbf{k}\cdot\mathbf{U})^2}{\rho(n+i\mathbf{k}\cdot\mathbf{U})^2+\mu H^2/4\pi}-2\rho\Omega^+D(k_x V-k_y U)\right] \\ &\quad\times\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2(n^*-i\mathbf{k}\cdot\mathbf{U})\,dz \\ &\quad-\int\{\tfrac{1}{4}\rho(\mathbf{k}\cdot D\mathbf{U})^2+[gk^2+2i\Omega^-(n+i\mathbf{k}\cdot\mathbf{U})]D\rho\} \\ &\quad\times\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2(n^*-i\mathbf{k}\cdot\mathbf{U})\,dz \\ &\quad+\int\frac{\mu H^2}{4\pi}\left[\frac{1}{2}\frac{i\mathbf{k}\cdot D^2\mathbf{U}}{(n+i\mathbf{k}\cdot\mathbf{U})}+\frac{3}{4}\frac{(\mathbf{k}\cdot D\mathbf{U})^2}{(n+i\mathbf{k}\cdot\mathbf{U})^2}\right] \\ &\quad\times\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2(n^*-i\mathbf{k}\cdot\mathbf{U})\,dz=0 \end{aligned} \tag{45}$$

Let us suppose that the real and imaginary parts of n are R and I respectively. Then, after dividing by $R > 0$, the real part of (45) is

$$\begin{aligned}
&\int \left[\rho + \frac{\mu H^2/4\pi}{|n+i\mathbf{k}\cdot\mathbf{U}|^2}\right]\left(|Df|^2 + k^2|f|^2\right) dz \\
&+ \int \frac{4\Omega^{+2}\rho^2(\rho|n+i\mathbf{k}\cdot\mathbf{U}|^2 + \mu H^2/4\pi)|n+i\mathbf{k}\cdot\mathbf{U}|^2}{|\rho(n+i\mathbf{k}\cdot\mathbf{U})^2 + \mu H^2/4\pi|^2}\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz \\
&- \int 2\rho\Omega^{+} D(k_x V - k_y U)\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz \\
&- \int [\tfrac{1}{4}\rho(\mathbf{k}\cdot D\mathbf{U})^2 + gk^2 D\rho]\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz \\
&+ \int \frac{\mu H^2}{4\pi}\frac{(I+\mathbf{k}\cdot\mathbf{U})\,\mathbf{k}\cdot D^2\mathbf{U}}{|n+i\mathbf{k}\cdot\mathbf{U}|^2}\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz \\
&+ \int \frac{\mu H^2}{4\pi}\frac{3}{4}\frac{[R^2 - 3(I+\mathbf{k}\cdot\mathbf{U})^2]\,(\mathbf{k}\cdot D\mathbf{U})^2}{|n+i\mathbf{k}\cdot\mathbf{U}|^4}\left|\frac{f}{n+i\mathbf{k}\cdot\mathbf{U}}\right|^2 dz = 0
\end{aligned} \tag{46}$$

Following the arguments given by Johnson (1963), this condition is satisfied if the sum of the last five integrals on the left-hand side is negative. On the contrary, if the sum is positive, this is not possible, and a sufficient condition for stability is

$$\begin{aligned}
&\frac{4\Omega^{+2}\rho^2(\rho|n+i\mathbf{k}\cdot\mathbf{U}|^2 + \mu H^2/4\pi)\,|n+i\mathbf{k}\cdot\mathbf{U}|^2}{|\rho(n+i\mathbf{k}\cdot\mathbf{U})^2 + \mu H^2/4\pi|^2} - 2\rho\Omega^{+}D(k_x V - k_y U) - \tfrac{1}{4}\rho(\mathbf{k}\cdot D\mathbf{U})^2 \\
&- gk^2 D\rho + \mu H^2/4\pi\left[\frac{(I+\mathbf{k}\cdot\mathbf{U})\,\mathbf{k}\cdot D^2\mathbf{U}}{|n+i\mathbf{k}\cdot\mathbf{U}|^2} + \frac{3}{4}\frac{[R^2 - 3(I+\mathbf{k}\cdot\mathbf{U})^2]\,(\mathbf{k}\cdot D\mathbf{U})^2}{|n+i\mathbf{k}\cdot\mathbf{U}]^4}\right] \geqslant 0.
\end{aligned} \tag{47}$$

It is clear from (47) that it is not possible to determine, in a definite way, the correct sign of the last term on the left-hand side, corresponding to the magnetic field effect. Moreover, it is not possible to know the relative magnitude of R and I.

Therefore a new method has been developed in which we are able to determine the effect of a horizontal magnetic field on the stability of a rotating stratified fluid under shear flow.

We start by substituting the z component of velocity w by $h(n + i\mathbf{k}\cdot\mathbf{U})$ in (16). Then, after multiplying by the complex conjugate of h, h^*, integrating over the range of z, and some algebra, we are led to

$$\begin{aligned}
&\int \left[\rho(n+i\mathbf{k}\cdot\mathbf{U})^2 + \frac{\mu H^2}{4\pi}\right](|Dh|^2 + k^2|h|^2)\,dz + \int \frac{4\Omega^{+2}\rho^2(n+i\mathbf{k}\cdot\mathbf{U})^2}{\rho(n+i\mathbf{k}\cdot\mathbf{U})^2 + \mu H^2/4\pi}|h|^2\,dz \\
&\quad - \int 2\rho\Omega^{+}D(k_x V - k_y U)\,|h|^2\,dz - \int [gk^2 + 2i\Omega^{-}(n+i\mathbf{k}\cdot\mathbf{U})]\,D\rho|h|^2\,dz = 0.
\end{aligned} \tag{48}$$

The corresponding real part is

$$R^2\left[\int \rho(|Dh|^2+k^2|h|^2)\,dz+\int \frac{4\Omega^{+2}\mu H^2/4\pi}{\left|(n+i\mathbf{k}\cdot\mathbf{U})^2+\dfrac{\mu}{4\pi}\dfrac{H^2}{\rho}\right|^2}|h|^2\,dz\right]$$

$$-\int \rho\left[(I+\mathbf{k}\cdot\mathbf{U})^2-\frac{\mu}{4\pi}\frac{H^2}{\rho}\right](|Dh|^2+k^2|h|^2)\,dz$$

$$+\int \frac{4\Omega^{+2}\rho\left[|n+i\mathbf{k}\cdot\mathbf{U}|^4-\dfrac{\mu}{4\pi}\dfrac{H^2}{\rho}(I+\mathbf{k}\cdot\mathbf{U})^2\right]}{\left|(n+i\mathbf{k}\cdot\mathbf{U})^2+\dfrac{\mu}{4\pi}\dfrac{H^2}{\rho}\right|^2}|h|^2\,dz$$

$$-\int 2\rho\Omega^{+}D(k_x V-k_y U)\,|h|^2\,dz-\int [gk^2-2\Omega^{-}(I+\mathbf{k}\cdot\mathbf{U})]\,D\rho|h|^2\,dz=0. \quad (49)$$

In order to satisfy this equation, the last four integrals must add to give a negative number. On the contrary, if the sum is positive, this is not possible, and a sufficient condition for stability is

$$\frac{4\Omega^{+2}\rho\left[|n+i\mathbf{k}\cdot\mathbf{U}|^4-\dfrac{\mu}{4\pi}\dfrac{H^2}{\rho}(I+\mathbf{k}\cdot\mathbf{U})^2\right]}{\left|(n+i\mathbf{k}\cdot\mathbf{U})^2+\dfrac{\mu}{4\pi}\dfrac{H^2}{\rho}\right|^2}$$

$$+2\rho\Omega^{+}D(k_y U-k_x V)+2\Omega^{-}(I+\mathbf{k}\cdot\mathbf{U})\,D\rho-gk^2D\rho$$

$$-\rho\left[(I+\mathbf{k}\cdot\mathbf{U})^2-\frac{\mu}{4\pi}\frac{H^2}{\rho}\right](\gamma^2+k^2)\geqslant 0, \quad (50)$$

where γ^{-1} is a representative length scale along the z direction. This inequality (50) will allow us to discuss the influence of the horizontal magnetic field. Note that from the imaginary part of (48), an equation for I, the frequency of the perturbation, may be obtained to determine the possible sign it may have for different magnitudes of the fields and wavenumber. We omit these calculations here.

3.1. Discussion of results related to the generalization of Johnson's approximation

When the horizontal stratified magnetic field is zero, the first three terms of the left-hand side of (47) correspond to those obtained by Johnson (1963). The last two terms of the left-hand side result from the density stratification and the magnetic field. We cannot determine the effect of the magnetic field with this inequality, as explained above. Then let us suppose that H is zero. Besides, without loss of generality, suppose that only Ω_x is different from zero, and therefore $\Omega^{+}=k_x\Omega_x$ and $\Omega^{-}=k_y\Omega_x$.

The value of Ω_x that gives a sufficient condition for stability can be found by solving the quadratic inequality (47). Thus

$$4\Omega_x \leqslant \frac{1}{k_x} D(k_x V - k_y U) - \frac{k}{k_x}\left[(DU)^2 + (DV)^2 + \frac{4gD\rho}{\rho}\right]^{1/2}, \tag{51}$$

$$4\Omega_x \geqslant \frac{1}{k_x} D(k_x V - k_y U) + \frac{k}{k_x}\left[(DU)^2 + (DV)^2 + \frac{4gD\rho}{\rho}\right]^{1/2}. \tag{52}$$

Note that

$$[D(k_x V - k_y U)]^2 + (\mathbf{k} \cdot D\mathbf{U})^2 = k^2[(DU)^2 + (DV)^2].$$

Now suppose that the velocity represents a Couette flow or has a hyperbolic-tangent profile with a maximum for its derivative $DU_{\max} = 1$ and that the density has an exponential stratification as $\rho = \rho_b e^{\beta z}$. Then a positive β broadens the unstable region of Fig. 2 of Johnson (1963). A negative β narrows this region until it becomes a line when $(DU)^2 + (DV)^2 = 4g|\beta|$. The magnitude of the wavenumber is important to determine the size of the unstable region. When k_y is zero, the wavenumber disappears. If k_x is negative, the inequalities must be changed. When $k_y \neq 0$, a small wavenumber $k_x > 0$ increases the unstable area.

3.2. *Discussion of results obtained from the new method*

Suppose again, without loss of generality, that only Ω_x is different from zero. Then, from the new method developed in this paper, the following solutions of the quadratic inequality (50) for Ω_x are obtained:

$$4\Omega_x \leqslant \frac{-R_1}{k_x^2 B_1} - \frac{1}{k_x^2 B_1}\left[R_1^2 + 4k_x^2 B_1\left(B_2 + \frac{gk^2 D\rho}{\rho}\right)\right]^{1/2}, \tag{53}$$

$$4\Omega_x \geqslant \frac{-R_1}{k_x^2 B_1} + \frac{1}{k_x^2 B_1}\left[R_1^2 + 4k_x^2 B_1\left(B_1 + \frac{gk^2 D\rho}{\rho}\right)\right\}^{1/2}, \tag{54}$$

where

$$B_1 = \frac{\left[|n + i\mathbf{k}\cdot\mathbf{U}|^4 - \frac{\mu}{4\pi}\frac{H^2}{\rho}(I + \mathbf{k}\cdot\mathbf{U})^2\right]}{\left|(n + i\mathbf{k}\cdot\mathbf{U})^2 + \frac{\mu}{4\pi}\frac{H^2}{\rho}\right|^2}, \tag{55}$$

$$B_2 = \left[(I + \mathbf{k}\cdot\mathbf{U})^2 - \frac{\mu}{4\pi}\frac{H^2}{\rho}\right](\gamma^2 + k^2), \tag{56}$$

$$R_1 = k_x D(k_y U - k_x V) + k_y (I + \mathbf{k}\cdot\mathbf{U})\frac{D\rho}{\rho}. \tag{57}$$

Before discussing these inequalities, we obtain from (50) the sufficient condition for stability when rotation is zero, namely

$$-\frac{gk^2 D\rho}{\rho} - \left[(I + \mathbf{k}\cdot\mathbf{U})^2 - \frac{\mu}{4\pi}\frac{H^2}{\rho}\right](\gamma^2 + k^2) \geqslant 0. \tag{58}$$

Note that this condition is similar to that of the two-fluid system presented by Chandrasekhar (1961, chap. XI (204), p. 511 – see the radicand). Here we may suppose that the density grows as $e^{\beta z}$ and the magnetic field as $e^{\beta z/2}$ and take a representative value for the velocity (its maximum, for example, equal to one in the case of a hyperbolic-tangent velocity profile). In this case, a positive β counteracts the role played by the magnetic field in stabilizing by eliminating the destabilizing effects of shear flow and the adverse density stratification. When β is negative, a weaker magnetic field will be enough to stabilize the system.

In order to understand the influence on stability of each of the terms in the inequalities (53) and (54), we start again with rotation and stratification alone. Suppose that $D(k_y U - k_x V) > 0$. In this case, $B_1 = 1$, $B_2 = (I + \mathbf{k}\cdot\mathbf{U})^2(\gamma^2 + k^2)$, R_1 is the same, and the inequalities (53) and (54) reduce to

$$4\Omega_x \leqslant \frac{-R_1}{k_x^2} - \frac{1}{k_x^2}\left\{R_1^2 + 4k_x^2\left[(I+\mathbf{k}\cdot\mathbf{U})^2(\gamma^2+k^2) + \frac{gk^2 D\rho}{\rho}\right]\right\}^{1/2}, \tag{59}$$

$$4\Omega_x \geqslant \frac{-R_1}{k_x^2} + \frac{1}{k_x^2}\left\{R_1^2 + 4k_x^2\left[(I+\mathbf{k}\cdot\mathbf{U})^2(\gamma^2+k^2) + \frac{gk^2 D\rho}{\rho}\right]\right\}^{1/2}. \tag{60}$$

Here again we shall refer to the unstable area presented in Johnson (1963, Fig. 2). Note that the conclusions are the same as those obtained above for the generalization of Johnson's results in Sect. 3.1.

The behaviour of the magnetic field, when interacting with rotation, may be better understood in the absence of density stratification. In this case, $R_1 = k_x D(k_y U - k_x V)$, B_1 and B_2 are the same, and the inequalities (53) and (54) become

$$4\Omega_x \leqslant \frac{-D(k_y U - k_x V)}{k_x B_1} - \frac{1}{k_x B_1}\{[D(k_y U - k_x V)]^2 + 4B_1 B_2\}^{1/2}, \tag{61}$$

$$4\Omega_x \geqslant \frac{-D(k_y U - k_x V)}{k_x B_1} + \frac{1}{k_x B_1}\{[D(k_y U - k_x V)]^2 + 4B_1 B_2\}^{1/2}. \tag{62}$$

Note first that if both are positive then $B_1 > B_2$. Then, if the magnetic field is increased to work against the destabilizing effect of shear flow, B_2 may become zero before B_1, and reduce the unstable area presented by Johnson. When $B_2 = 0$, one of the roots becomes zero (see (50)). A further increase in the magnetic field reduces the unstable area even more until $B_1 = 0$ and only one root exists (see (50)), which is equal to:

$$\Omega_x \geqslant \frac{-\left[\dfrac{\mu}{4\pi}\dfrac{H^2}{\rho} - (I+\mathbf{k}\cdot\mathbf{U})^2\right]}{2k_x D(k_y U - k_x V)}, \tag{63}$$

where we have already supposed that the denominator is positive and, for this magnitude of magnetic field, the numerator is negative. This means that the unstable area of Johnson has grown infinitely to the left of this value and that it is only possible to stabilize for values above this one.

A further increase in the magnetic field makes both $B_1 < 0$ and $B_2 < 0$, and the two roots are recovered, again reducing the unstable area from an infinitely large one to a finite one. However, note that when this occurs, the inequalities (61) and (62) interchange roles, because the denominator has already become negative. Increasing the magnetic field even more increases the unstable area by increasing the magnitude of the radicand.

The description given above does not change when density stratification is included, as in (53) and (54). The difference is that we need to think about the sign of $B_1 + gk^2 D\rho/\rho$ instead of the sign of B_2 alone. However, for large values of β, it is possible for this term to become negative after B_1 does. For negative β, it becomes negative for smaller magnetic fields.

4. Conclusions

We have investigated the KHI of a non-dissipative incompressible magnetofluid under the action of horizontal rotation and a stratified horizontal magnetic field. The maximum growth rate of the instability has been obtained, along with an estimate of its upper bound. By means of the maximum growth rate, we have given sufficient conditions for stability. We have shown that there are terms that only appear in the presence of shear flow compared with the RTI. One term with magnetic field, $-\sigma_1^2$, already appears in RTI, while others appear owing to the presence of shear flow. There is a term, $4\Omega^{+2} E_4'^2/k^2$, that needs the simultaneous action of rotation, magnetic field and shear to be present in the growth rate. Another term, $-2\Omega^- A/k^2$, discussed in Case 3 of Sec. 2, appears owing to the interaction of rotation, fluid stratification and shear.

A generalization of the sufficient condition for stability given by Johnson (1963) has been presented in Sec. 3 for the case of horizontal rotation and density stratification alone. We have shown how unstable density stratification broadens the unstable area of figure 2 of the paper by Johnson. Stable density stratification may reduce the unstable area to a straight line. We have also shown that Johnson's method is not adequate to understand the behaviour of the magnetic field. Therefore a new method has been developed to show the role played by the magnetic field when horizontal rotation, density stratification and shear flow are present. We have found that an increase in the magnetic field decreases the unstable area presented in figure 2 given by Johnson. A further increase, instead of reducing this area, increases it infinitely to the left of the figure, and it is only possible to stabilize for rotation values larger than the calculated one. This situation occurs when the equation has only one root at a specific magnitude of the magnetic field. If we increase the magnetic field even further, the equation again has two roots, and the unstable region becomes finite, but an increase in the magnetic field increases it once more. The inclusion of density stratification does not change the qualitative nature of the above results, but may only change the order at which B_1 becomes zero when the magnetic field is increased.

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