Superconducting transition-temperature enhancement due to electronic-band-structure density-of-states

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We briefly review a simple statistical model of a boson-fermion mixture of unpaired fermions plus linear-dispersion-relation Cooper pairs that leads to Bose-Einstein condensation (BEC) for all dimensions greater than unity. (The "dispersion relation" of a particle is its energy vs. momentum relation.) This contrasts sharply with "ordinary" BEC for a many-boson assembly of non-interacting bosons each moving in vacuum with a quadratic dispersion relation, which is well-known to occur only for dimensions greater than two. The BEC critical temperatures T_c are substantially higher than those of the BCS theory of superconductivity, for the same BCS model interaction between the fermions that gives rise to the Cooper pairs, at both weak and strong couplings. However, these results hold with an ideal-fermi-gas (IFG) density-of-states (DOS) for the underlying electron (or hole) carriers. We then show that even higher T_c values are obtained in 2D if a non-IFG DOS is employed which reflects the electronic band structure of the quasi-2D copper-oxygen planes characteristic of cuprate superconductors. The non-IFG DOS used are both a so-called Van Hove scenario (VHS) with a logarithmic singularity in the DOS, and a DOS with a power-law-singularity associated with an extended-saddle-point (ESP) in the energy-momentum curve.

Keywords: Superconductivity; Bose-Einstein condensation; electronic density of states

Recordamos brevemente un modelo estadístico simple de una mezcla de fermiones desapareados más pares de Cooper que satisfacen una relación de dispersión lineal que les permite condensarse a la Bose-Einstein (BEC) en dimensiones mayores que la unidad. (La relación de dispersión de un partícula es su energía en función de su momento.) Lo anterior contrasta notoriamente con la BEC "ordinaria" de un conjunto de muchos bosones que no interactúan entre sí, y moviéndose en el vacío con una relación de dispersión cuadrática, la cual es bien sabido se da sólo para dimensiones mayores que dos. Las temperaturas críticas T_c de la BEC son sustancialmente más elevadas que las de la teoría de la superconductividad BCS, para el mismo modelo de interacción BCS entre los fermiones que forman los pares de Cooper, en ambos límites de acoplamiento débil y fuerte. Sin embargo, estos resultados son para electrones (u hoyos) con una densidad de estados (DOS) igual a la de un gas ideal de fermiones (IFG). Mostramos que T_c más altas aún son obtenidas en 2D si empleamos una DOS diferente de la IFG, que refleje la estructura de bandas electrónicas de los planos de cobre-oxígeno característicos de los cupratos superconductores. Las DOS diferentes de la IFG usadas aquí son la de el llamado escenario de Van Hove que contiene una singularidad logarítmica en la DOS y, una DOS con una singularidad potencial asociada a con un punto silla extendido (ESP) en la curva de energía *vs.* momento.

Descriptores: Superconductividad; condensación Bose-Einstein; densidad de estados electrónicos

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1. Introduction

Consider [1] first an ideal quantum gas in d dimensions of permanent (*i.e.*, number-conserving) bosons with a general dispersion relation

$$\varepsilon_k = C_s k^s, \qquad \text{with} \quad s > 0. \tag{1}$$

For ordinary bosons of mass m in vacuum $s = 2$ and $C_s = \hbar^2/2m$, while for a Cooper pair in the Fermi sea $s = 1$ as discussed below. The boson number density n_B in a "box" of length L in d dimensions is defined as $n_B \equiv N_B/L^d$, where the total number of bosons $N_B = N_{B,0}(T) + \sum_{k\neq 0}[e^{\beta(\varepsilon_k-\mu_B)}-1]^{-1}$, with $\mu_B \leq 0$ the chemical potential, $\beta \equiv 1/k_B T$, k_B the Boltzmann constant and T is the absolute temperature. The summation implies an elementary integral easily evaluated in terms of the usual Bose integrals [2] (with $z \equiv e^{\beta\mu_B}$ the fugacity)

$$g_\sigma(z) \equiv \frac{1}{\Gamma(\sigma)}\int_0^\infty dx \frac{x^{\sigma-1}}{z^{-1}e^x - 1} = \sum_{l=1}^{\infty}\frac{z^l}{l^\sigma} \xrightarrow[z\to 1]{} \zeta(\sigma). \tag{2}$$

The last identification holds when $\sigma \geq 1$, where $\zeta(\sigma)$ is the Riemann Zeta-function of order σ. The function $\zeta(\sigma) < \infty$ for $\sigma > 1$, while the series $g_\sigma(1)$ diverges for $\sigma \leq 1$. As T is lowered down to T_c, below which $N_{B,0}(T)$ just ceases to be negligible compared with N_B and simultaneously $\mu_B \simeq 0^-$. The condensate fraction for $0 < T < T_c$ in d dimensions is the fractional number $N_{B,0}(T)/N_{B,0}(0)$ of bosons in the $k = 0$ state. Note that $\mu_B \simeq 0^-$ over the entire temperature range $0 < T < T_c$ since $N_{B,0}(T) = (e^{-\beta\mu_B} - 1)^{-1}$

implies that $e^{\beta\mu_B} = N_{B,0}(T)/[N_{B,0}(T)+1] < 1$, and approaches 1^- over this *entire* temperature range because $N_{B,0}(T)$ on cooling grows to a sizeable fraction of N_B which is macroscopic. Since $N_B = N_{B,0}(T) + N_{B,k>0}(T)$ while $N_B = N_{B,0}(0)$, for $d > 0$ one can write the fractional number as

$$\frac{N_{B,0}(T)}{N_{B,0}(0)} = 1 - \left[2^{d-1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)n_B\right]^{-1} \times \int_0^\infty \frac{dk\, k^{d-1}}{e^{\beta C_s k^s}-1}, \quad (3)$$

where the sum-to-integral conversion [2]

$$\sum_{\mathbf{k}\neq 0} \longrightarrow \left[\frac{2\pi^{d/2}}{\Gamma(d/2)}\right]\left(\frac{L}{2\pi}\right)^d \int dk\, k^{d-1}$$

was employed. Using (2) to evaluate the integral in (3) gives

$$\frac{N_{B,0}(T)}{N_{B,0}(0)} = 1 - \left[2^{d-1}\pi^{d/2}\Gamma\left(\frac{d}{2}\right)n_B\right]^{-1} \times \frac{\Gamma(d/s)\, g_{d/s}(1)}{s(\beta C_s)^{d/s}}. \quad (4)$$

Since this is negligible at $T = T_c$, setting the rhs of (4) to zero gives a simple algebraic equation whose solution yields the *general T_c formula*

$$T_c = \frac{C_s}{k_B}\left[\frac{s\,(2\pi)^d\,\Gamma(d/2)}{2\pi^{d/2}\,\Gamma(d/s)\,g_{d/s}(1)}n_B\right]^{s/d}. \quad (5)$$

The most familiar special case is for $d = 3$ and $s = 2$ which gives the well-known formula for the Bose-Einstein condensation (BEC) transition temperature

$$T_c = \frac{2\pi\hbar^2 n_B^{2/3}}{mk_B[\zeta(3/2)]^{2/3}} \simeq \frac{3.31\hbar^2 n_B^{2/3}}{mk_B}, \quad (6)$$

where in the last step we used $\zeta(3/2) \simeq 2.612$.

2. Ideal fermion gas density of states

Consider now a many-fermion system with attractive interactions capable of pairing at least *some* of the fermions into Cooper pairs (or "pairons"); these are considered as "bosons" even through they do not obey Bose commutation relations since they *do obey* a Bose-Einstein distribution [8] if these pairs are of definite center-of-mass (but *not* definite relative) momenta.

2.1. BCS Theory T_c

The BCS T_c formula is the solution of $\Delta(T_c) = 0$, where $\Delta(T)$ is the BCS energy gap, which in turn is the solution of the BCS gap equation [3]. The well-known results is

$$T_c \underset{\lambda\to 0}{\longrightarrow} 1.13\Theta_D\, e^{-1/\lambda} \quad (7)$$

where $\Theta_D \equiv \hbar\omega_D/k_B$ is the Debye temperature, the dimensionless coupling constant $\lambda \equiv g(E_F)V$. Here $g(E_F)$ is the density of one-spin, electronic states (DOS) at the Fermi surface with energy E_F, and V is the attractive strength of the BCS model interaction,

$$V_{\mathbf{k},\mathbf{k}'} = \begin{cases} -V & \text{if} \quad E_F - \hbar\omega_D \le \dfrac{\hbar^2k^2}{2m}, \\ & \qquad \dfrac{\hbar^2k'^2}{2m} \le E_F + \hbar\omega_D \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

where $V_{\mathbf{k},\mathbf{k}'}$ is the double Fourier transform of the space-dependent pair interaction between fermions, and $\mathbf{k}$, $\mathbf{k}'$ are *relative* wavevectors. The positive coupling constant V represents the net attractive effect of the electron-phonon interaction which overwhelms the repulsive Coulomb interaction.

2.2. BEC theory T_c for linear-dispersion-relation Cooper pairs

In d dimensions the number of fermions is $N = 2\sum_{\mathbf{k}}\theta(k_F - k)$, where $\theta(x)$ is the Heaviside step function, so that using the sum-to-integral conversion again the *fermion* number density n becomes

$$n \equiv \frac{N}{L^d} = \frac{k_F^d}{2^{d-2}\pi^{d/2}d\,\Gamma(d/2)}. \quad (9)$$

On the other hand, the number of bosons $N_{B,0}(0)$ actually formed at $T = 0$ through the BCS model interaction in the fermion gas is precisely $N_{B,0}(0) = g(E_F)\hbar\omega_D$, where [4]

$$g(\varepsilon) \equiv \left(\frac{L}{2\pi}\right)^d \frac{d^dk}{d\varepsilon} = \left(\frac{m^*}{2\pi\hbar^2}\right)^{d/2}\frac{L^d\varepsilon^{d/2-1}}{\Gamma(d/2)}, \quad (10)$$

since $g(\varepsilon)$ is the number of up-spin fermion states per unit energy. If *all* fermions were imagined paired, then $n_B/n = 1/2$. However, since $n_B = N_{B,0}(0)/L^d = g(E_F)\hbar\omega_D/L^d$, (9) and (10) show that in fact

$$\frac{n_B}{n} = \frac{d\hbar\omega_D}{4E_F} \equiv \frac{\nu d}{4}, \quad (11)$$

a fraction much less than 1/2 since typically $\hbar\omega_D \ll E_F$ or $\nu \ll 1$. This allows writing (5) with $s = 1$ as

$$\frac{T_c}{T_F} = \frac{a(d)\,\hbar v_F}{k_B}\left[\frac{\pi^{\frac{d+1}{2}}\,n_B}{\Gamma(\frac{d+1}{2})g_d(1)}\right]^{1/d} = 2a(d)\left[\frac{\nu}{2\Gamma(d)\zeta(d)}\right]^{1/d}. \quad (12)$$

for *a pure unbreakable-pairon gas*. In particular, for $d = 2$ and 3, with $a(2) = 2/\pi$ and $a(3) = 1/2$ [5], and $n_B/n = \nu/2$ and $3\nu/4$, respectively, (12) gives

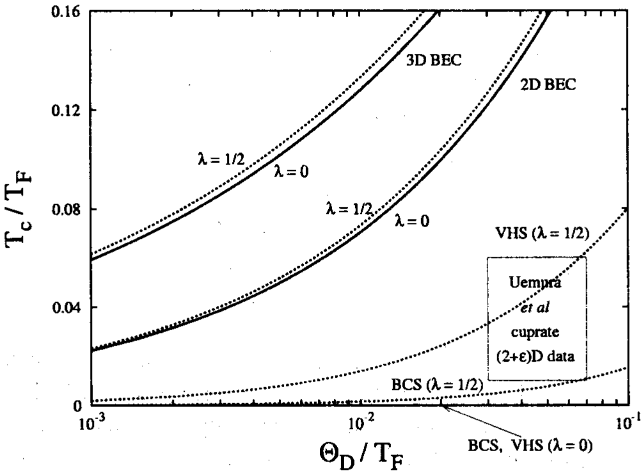

FIGURE 1. Scaled critical BEC temperature T_c/T_F *vs.* Θ_D/T_F for the simple boson-fermion model in 2D and 3D, Eq. (13) and as discussed in text. Also shown is the maximal BCS result $T_c = 1.13\,\Theta_D\, e^{-1/\lambda}$, as well as the "maximal" Van Hove scenario (VHS) result obtained using (16) for a logarithmically-singular DOS, both for $\lambda = 1/2$. Both BCS and VHS T_c values vanish as $\lambda \to 0$, albeit at exponentially different rates. The rectangle comprises empirical data for the 2D-like cuprate superconductors [7].

$$\frac{T_c}{T_F} = \begin{cases} \left(\frac{4\sqrt{6}}{\pi^2}\right)\sqrt{\frac{n_B}{n}} = \\ \qquad \left(\frac{4\sqrt{3}}{\pi^2}\right)\sqrt{\nu} \simeq 0.702\sqrt{\nu} \qquad \text{(2D)} \\ \frac{(n_B/n)^{1/3}}{[3\zeta(3)]^{1/3}} = \\ \qquad \left[\frac{\nu}{4}\zeta(3)\right]^{1/3} \simeq 0.592\nu^{1/3} \qquad \text{(3D)}. \end{cases} \quad (13)$$

These results hold for arbitrarily weak coupling, but are remarkably insensitive for higher coupling [9], *increasing* by only about 4% for λ as large as 1/2, in either 2D and 3D for $\nu = 0.05$ and 0.001, respectively, see Fig. 1.

It is a fact [6] that Cooper pairs break up beyond a certain center-of-mass momentum (CMM). For *breakable* pairons $T_c \to \infty$ as $\lambda \to 0$ [8] since the (finite) upper limit in (3) turns out to be proportional to the Cooper-pair binding energy which vanishes when $\lambda \to 0$, meaning that *all* bosons are in the $K = 0$ at *any* temperature. It is a remarkable result that mixing into this ideal gas of breakable bosons the unpaired fermions then gives a *finite* T_c with very minor corrections to (13) [9].

3. Van Hove Scenario Density of States

3.1. Exact BCS T_c formula in VHS scenario

We first recall the BCS T_c formula within the VHS approach to superconductivity. The finite-temperature gap-energy $\Delta(T)$ equation in BCS theory with a general density-of-states

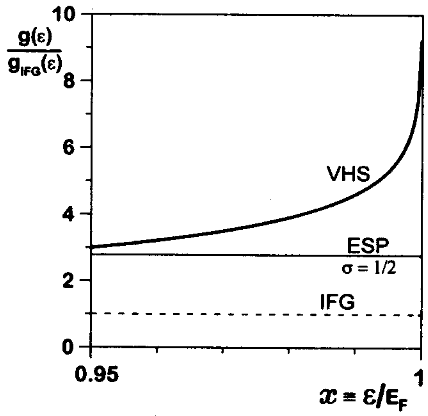

FIGURE 2. VHS and ESP DOS, normalized to the IFG DOS, as explained in text, is the vicinity of x just below 1 relevant to the $\nu = 0.05$ cuprate typical value of Θ_D/T_F.

(DOS) $g(\varepsilon)$, namely

$$\frac{2}{V} = \int_{E_F-\hbar\omega_D}^{E_F+\hbar\omega_D} \frac{g(\varepsilon)d\varepsilon}{\sqrt{(\varepsilon-E_F)^2+\Delta^2(T)}} \times \tanh\frac{\sqrt{(\varepsilon-E_F)^2+\Delta^2(T)}}{2k_BT_c}. \quad (14)$$

By (8) V is nonzero only in a narrow shell of thickness $2\hbar\omega_D$ centered about the Fermi energy E_F. Assuming a Van Hove singularity (VHS) DOS of the form

$$g(\varepsilon) \equiv g(E_F)\ln\left|\frac{E_F}{\varepsilon-E_F}\right|, \quad (15)$$

see Fig. 2, Tsuei *et al.* [10] obtain an *approximate* BCS T_c-formula from (14) and (15) from the condition $\Delta(T_c) \equiv 0$, by approximating $\tanh x \simeq x$ $(x<1)$, $\tanh x \simeq 1$ $(x>1)$ in the gap equation. However, an *exact* T_c formula was derived [11] as the solution of the transcendental equation

$$\frac{T_c}{T_F} = \frac{1}{2}\exp\left\{-\left[\left(\frac{1}{g(E_F)V}+D\left(\frac{\Theta_D}{2T_c},\frac{T_F}{2T_c}\right)\right) \times 2\coth\frac{\Theta_D}{2T_c}+\ln^2\frac{T_F}{\Theta_D}\right]^{1/2}\right\}. \quad (16)$$

where

$$D(Z,W) \equiv \int_0^Z dx\left[\ln x\,\ln\frac{W}{x}+\frac{1}{2}\ln^2 x\right]\mathrm{sech}^2 x. \quad (17)$$

This was found to give T_c values roughly 15% *smaller* than (3) of [10], for given $\lambda \equiv g(E_F)V$, Θ_D and T_F. Fig. 1 shows a plot of (16) for $\lambda = 1/2$, marked VHS; an enhancement of about tenfold over the BCS result (7) is evident. Both BCS T_c values, with IFG and VHS DOS results vanish as $\lambda \to 0$ at different rates.

3.2. BEC T_c in the VHS

Recalling (13) for $2D$, as well as (9) and $n_B \equiv N_{B,0}(0)/L^d = g(E_F)\hbar\omega_D/L^d$, if $N_{2,0}(0) = 2N_{B,0}(0)$, where $N_{2,0}(0)$ is the number of paired fermions at $T = 0$, the number of pairable fermions $N_2(T)_{\rm VHS}$ becomes

$$N_2(T)_{\rm VHS} = 2g(E_F)\int_{\mu-\hbar\omega_D}^{\mu+\hbar\omega_D} d\varepsilon \frac{\ln\left|\frac{E_F}{\varepsilon - E_F}\right|}{e^{\beta(\varepsilon-\mu)}+1}. \quad (18)$$

where the VHS DOS has been substituted in the last equation. Letting $x \equiv \varepsilon/E_F$, $\tilde{\mu} \equiv \mu/E_F$, $\tilde{\beta} \equiv \beta E_F$ and since $\nu \equiv \hbar\omega_D/E_F$, (18) reduces to

$$N_2(T)_{\rm VHS} = -2g(E_F)E_F\int_{\tilde{\mu}-\nu}^{\tilde{\mu}+\nu} dx \frac{\ln|x-1|}{e^{\tilde{\beta}(x-\tilde{\mu})}+1}. \quad (19)$$

Introducing the new integration variable $y = x - 1$, we have

$$N_2(T)_{\rm VHS} = -\,2g(E_F)E_F \int_{\tilde{\mu}-\nu-1}^{\tilde{\mu}+\nu-1} dy \frac{\ln|y|}{e^{\tilde{\beta}(y+1-\tilde{\mu})}+1}. \quad (20)$$

When $T \to 0$, $\tilde{\beta} \to \infty$ and $\mu = E_F$, or $\tilde{\mu} = 1$, so that (20) becomes

$$N_2(T)_{\rm VHS} \xrightarrow[T\to 0]{} 0$$

$$N_{2,0}(0)_{\rm VHS} = -2g(E_F)E_F\int_{-\nu}^{\nu} dy\theta(\tilde{\mu}-y)\ln|y|$$

$$= -2g(E_F)E_F\int_{-\nu}^{0} dy \ln|y| \quad (21)$$

$$= 2g(E_F)E_F(\nu - \nu\ln\nu) = 2N_{B,0}(0). \quad (22)$$

Consequently

$$n_B \equiv N_{B,0}(0)/L^2 = \frac{g(E_F)E_F(\nu-\nu\ln\nu)}{L^2}, \quad (23)$$

so that recalling (9) and (10) this gives

$$\frac{n_B}{n} = \frac{mL^2E_F(\nu-\nu\ln\nu)2\pi}{2\pi\hbar^2k_F^2L^2} \quad (24)$$

where $k_F^2 = 2mE_F/\hbar^2$, which substituted into (24) leaves

$$\frac{n_B}{n} = \frac{\nu - \nu\ln\nu}{2}. \quad (25)$$

Finally, using (13) we have

$$\frac{T_c^{\rm VHS}}{T_F} = \sqrt{1-\ln\nu}\,\frac{4\sqrt{3}}{\pi^2}\sqrt{\nu} \simeq 2\frac{T_c^{\rm IFG}}{T_F}, \quad (26)$$

i.e., an enhancement of roughly two for $\nu = 0.05$ which is typical for cuprates.

4. Extended-saddle-point singularity DOS

We now consider an ESP energy dispersion curve suggesting a power-law (as opposed to a logarithmic) singularity DOS [12]. These dispersion curves have actually been observed in angle-resolved photoemission experiments [13] in such superconductors like $YBa_2Cu_3O_{6.9}$ and $YBa_2Cu_4O_8$.

Since $x \equiv \varepsilon/E_F$, we write this DOS as

$$g(\varepsilon) = \frac{N}{2E_F}f(x) \quad (27)$$

with the dimensionless function $f(x)$ defined by

$$f(x) = \frac{x}{(1-x)^\sigma}\theta(1-x_0-x) + f_0\theta(x+x_0-1)\theta(1-x), \qquad \sigma > 0, \quad (28)$$

where $\theta(y) = 0$ if $y < 0$, $= 1$ if $y > 0$ and $= 1/2$ if $y = 0$. The curve $f(x)$ rises linearly from zero at $x = 0$ and would diverge at $x = 1$ with a power-σ-singularity; this rise is then interrupted at $x = 1 - x_0$ beyond which it flattens out into a "plateau" of height f_0 and width x_0. The value of f_0 is determined by

$$\int_0^1 f(x)\,dx = \int_0^{1-x_0}\frac{x}{(1-x)^\sigma}dx + \int_{1-x_0}^1 f_0\,dx = 1, \quad (29)$$

which is a (normalization) condition ensuring that the total number of fermions at $T = 0$ is $N = \int_0^\infty d\varepsilon\, g(\varepsilon)\theta(E_F-\varepsilon)$. Note that from (29) f_0 is expressible solely in terms of x_0 and σ, specifically

$$f_0 = x_0^{-1} - \frac{(x_0)^{\sigma-1} + x_0(1-\sigma) + (\sigma-2)}{(2-3\sigma+\sigma^2)x_0^\sigma}. \quad (30)$$

Further, as $\sigma \to \infty$ and $x_0 \to 1$, $f_0 \to 1$ and one recovers the IFG case. However, as $\sigma \to 0$ we find that there is a minimum value $\sigma_{\rm min} \simeq 0.3819$ beyond which x_0 ceases to be real. The exponent $\sigma = 1/2$ is suggested in [12] on physical grounds.

4.1. BCS T_c in ESP singularity

Consider the gap equation for the finite-temperature energy gap $\Delta(T)$ (14) but with an ESP DOS of the form (27), where the dimensionless function $f(x)$ satisfies (28) through (30), except that (28) is modified to be left-right symmetric about $x = 1$ as appropriate if $T \geq 0$. See Fig. 2 for case $\sigma = 1/2$. Thus the BCS gap equation becomes, where again $x \equiv \varepsilon/E_F$,

$$\frac{2}{V} = N\int_{1-\nu}^{1+\nu} dx \frac{f(x)}{\sqrt{[E_F(x-1)]^2+\Delta^2(T)}} \times \tanh\frac{\sqrt{[E_F(x-1)]^2+\Delta^2(T)}}{2k_BT_c}, \quad (31)$$

where we have used the symmetry of the integrand about E_F. To determine T_c one must solve $\Delta(T_c) = 0$; this reduces (31) to

$$\frac{2}{V} = \frac{N}{E_F} \int_{1-\nu}^{1+\nu} dx \frac{f(x)}{x-1} \tanh x \, \frac{x-1}{2\tilde{T}_c}, \tag{32}$$

with $\tilde{T}_c \equiv T_c/T_F$ and (as before) $\nu \equiv \hbar\omega_D/E_F$ being dimensionless. The DOS rises from zero energy, and becomes a plateau at x equals $1 - x_0$; this plateau for $\sigma = 1/2$ is exhibited in Fig. 2 in the vicinity just below E_F.

For $x_0 > \nu$, $f(x) = f_0$, the constant given by (30). Introducing the new variable $z = (x-1)/2\tilde{T}_c$ we are then left with

$$\frac{E_F}{NVf_0} = \int_0^{\nu/2\tilde{T}_c} dz \frac{\tanh z}{z}, \tag{33}$$

since the integrand is even. Integrating by parts gives

$$\frac{E_F}{NVf_0} = \ln \frac{\nu}{2\tilde{T}_c} - \int_0^{\nu/2\tilde{T}_c} dz \ln z \, \text{sech}^2 z. \tag{34}$$

Assuming $T_F \gg T_c$ the upper limit can be replaced by infinity making the integral exact and equal to $-\ln 4e^{\gamma}/\pi$ [14]. In this case

$$\frac{E_F}{NVf_0} = \ln\left(\frac{\nu}{2}\tilde{T}_c\right) + \ln\left(\frac{4e^{\gamma}}{\pi}\right). \tag{35}$$

Since $g_{\text{IFG}}(E_F) = L^2 m/2\pi\hbar^2$, $\lambda \equiv V g_{\text{IFG}}(E_F)$, $E_F = \hbar^2 k_F^2/2m$ and $n = N/L^2 = k_F^2/2\pi$, one finally obtains

$$T_c = \frac{2\nu e^{\gamma}}{\pi} e^{-1/\lambda f_0} = 1.13\nu e^{-1/\lambda f_0}, \tag{36}$$

revealing only a slight enhancement in T_c for the ESP over that for the IFG DOS, since $f_0 \geq 1$ with equality in the IFG case.

4.2. BEC T_c in ESP singularity

We now determine the critical temperature T_c associated with BEC assuming a DOS resulting from an extended-saddle-point (ESP). For this we must recalculate n_B/n as in (11) and then insert this into the 2D version of (13). The number of pairable fermions is

$$N_2(T)_{\text{ESP}} = 2\int_{\mu-\hbar\omega_D}^{\mu+\hbar\omega_D} \frac{g(\varepsilon)}{\exp\beta(\varepsilon-\mu)+1} d\varepsilon, \tag{37}$$

where $g(\varepsilon)$ is defined by (27) and (28). Since for $x_0 > \nu$ the DOS $g(\varepsilon)$ is a constant over the entire integration range, one has

$$N_2(T)_{\text{ESP}} = \frac{Nf_0}{E_F}\int_{\mu-\hbar\omega_D}^{\mu+\hbar\omega_D} \frac{1}{\exp\beta(\varepsilon-\mu)+1} d\varepsilon$$

$$= Nf_0 \int_{\tilde{\mu}-\nu}^{\tilde{\mu}+\nu} \frac{1}{\exp\tilde{\beta}(x-\tilde{\mu})+1} dx, \tag{38}$$

where $\tilde{\beta} \equiv \beta E_F$, $\tilde{\mu} \equiv \mu/E_F$, and as before $x \equiv \varepsilon/E_F$ while $\nu \equiv \hbar\omega_D/E_F$. If $T = 0$, $\tilde{\mu} = 1$ and (38) becomes

$$N_2(0)_{\text{ESP}} = Nf_0 \int_{1-\nu}^{1+\nu} \theta(1-x)dx$$

$$= Nf_0 \int_{1-\nu}^{1} dx = Nf_0\nu, \tag{39}$$

so that the boson number at $T = 0$ is just

$$N_B(0) = \frac{1}{2}N_2(0)_{\text{ESP}} = \frac{1}{2}Nf_0\nu. \tag{40}$$

Since $n_B = N_B(0)/L^2$, while $n = N/L^2$, the first member on the rhs of (13) for 2D finally gives

$$\frac{T_c^{\text{ESP}}}{T_F} = \sqrt{f_0}(4\sqrt{3}/\pi^2)\sqrt{\nu}$$

$$= \sqrt{f_0}\,\frac{T_c^{\text{IFG}}}{T_F} \qquad (x_0 > \nu), \tag{41}$$

or an enhancement by a factor $\sqrt{f_0}$ of T_c with the ESP DOS over that with the IFG DOS. On the other hand, if $x_0 < \nu$, then $g(\varepsilon)$ consists of two parts, since (27) and (28) in (37) for $T = 0$ gives

$$N_2(0) = N\int_{1-\nu}^{1-x_0} \frac{x}{(1-x)^{\sigma}}dx + Nf_0\int_{1-x_0}^{1} dx. \tag{42}$$

Thus, (40) and (42) leave

$$\frac{n_B}{n} \equiv \frac{N_B(0)}{N}$$

$$= \frac{1}{2}\int_{1-\nu}^{1-x_0} \frac{x}{(1-x)^{\sigma}}dx + f_0 x_0/2. \tag{43}$$

Finally, substituting (43) into (13) gives

$$\frac{T_c^{\text{ESP}}}{T_F} = \sqrt{\left(\frac{2}{\nu}\right)\left(\frac{n_B}{n}\right)}\left(\frac{4\sqrt{3}}{\pi^2}\right)\sqrt{\nu})$$

$$= \sqrt{\left(\frac{2}{\nu}\right)\left(\frac{n_B}{n}\right)}\frac{T_c^{\text{IFG}}}{T_F} \qquad (x_0 < \nu), \tag{44}$$

with enhancement factor $\sqrt{(2/\nu)(n_B/n)}$, where n_B/n must be determined by (43).

In Table I are listed several values of x_0 and f_0 for different σ values. In the last column we give T_c^{ESP}/T_c^{IFG}, the resulting enhancement factor for the ESP DOS. This is plotted in Fig. 3 for several values of the exponent σ, with the exponent $\sigma = 1/2$ proposed in [12] being marked in the figure.

TABLE I. The corresponding ESP DOS $g(\varepsilon)$ parameters in (27) and (28), x_0, f_0, for some choices of σ.

σ	x_0	f_0	$T_c^{\rm ESP}/T_c^{\rm IFG}$
0.3819	0.0	∞	2.23245
0.4	0.009754	6.31065	2.14177
0.5	0.1038	2.7816	1.66784
1.5	0.486122	1.51615	1.23132
10	0.82775	1.14071	1.06804

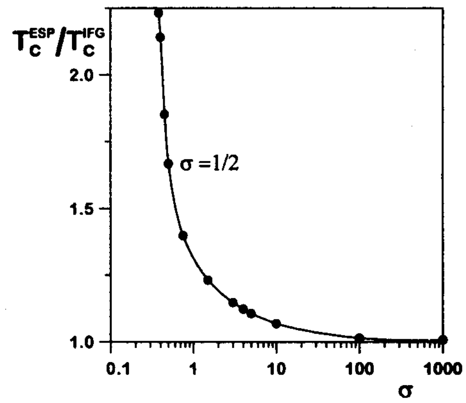

FIGURE 3. Enhancement of the BEC T_c using the ESP DOS, over that with the IFG DOS, for several values of the exponent σ in (27) and (28), the value $\sigma = 1/2$ being that suggested in [12].

5. Conclusions

Enhancement factors for the superconducting transition temperature T_c in a Bose-Einstein condensation (BEC) picture can be as high as about 32 for a strongly-coupled ($\lambda = 1/2$) Bardeen-Cooper-Schrieffer (BCS) model interaction, over that of the familiar BCS theory. Indeed, for weak coupling ($\lambda \to 0$) *this enhancement diverges*, meaning that BEC can arise even for arbitrarily-weakly-coupled Cooper pairs, which are bosons.

Much more moderate enhancements obtain in either the BCS or the BEC pictures when non-ideal-Fermi-gas (IFG) electronic-density-of-states (DOS) characteristic of the electronic-band-structure of quasi-2D systems like the cuprate superconductors are employed. Two such DOS's are the Van Hove scenario (VHS) logarithmically-divergent DOS, as well as the extended-saddle-point (ESP) power-law-divergence DOS.

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