A note on Newtonian and non-Newtonian oscillatory pipe flows

J. Ramón Herrera V.
Instituto Tecnológico de Zacatepec (ITZ-DGIT), Calzada Tecnológico
Apartado postal 45, No. 27, Zacatepec, Morelos, Mexico

B. Mena*
Instituto de Investigaciones en Materiales, Universidad Nacional Autónoma de México
Apartado postal 70-360, 04510 Ciudad Universitaria, México, D.F., Mexico

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In this note a simplified analysis is presented in which the flow of Newtonian and non-Newtonian liquids under a constant pressure gradient with an oscillatory motion of the boundaries is examined theoretically. The departure of the velocity function from the steady value due to the superimposed oscillations is analyzed for a linear viscoelastic constitutive equation in the case of rectilinear pipe flow and for flow between parallel planes. Since the viscosity function is not shear dependent, no increase in flow rate is possible; nevertheless, elastic effects modify the velocity profiles and need to be considered in order to fully explain the experimental increases in flow rate reported by other authors.

Keywords: Oscillatory flow; viscoelastic fluids

Examinamos teóricamente el flujo de líquidos newtonianos y no-newtonianos sometidos a un gradiente de presión constante y con movimiento oscilante en la frontera. Analizamos la desviación de la función velocidad, debida a la superposición de oscilaciones con respecto al flujo en estado permanente, para los casos de flujo en conductos de sección transversal circular y entre placas paralelas, para un fluido linealmente viscoelástico.Como la viscosidad no depende de la rapidez de deformación, no puede existir aumento en el gasto; sin embargo, los efectos elásticos modifican el perfil de velocidades y deben ser considerados en la explicación del aumento de gasto reportado por otros autores.

Descriptores: Flujo oscilante; fluidos viscoelásticos

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1. Introduction

For several decades, attention has been drawn to the flow of viscoelastic liquids under a constant pressure gradient in pipes which are oscillating longitudinally about a mean value [1–7]. This type of flow gives rise to substantial differences with respect to the flow of purely viscous liquids under similar circumstances, such as considerable increases in the flow rate when compared to purely rectilinear flow [4–10].

Theoretical studies of the superposition of longitudinal or transverse oscillations on steady flow in a circular uniform pipe have been carried out by Kazakia and Rivlin [4] for a third order fluid, by Mena *et al.* [5] for a general linear viscoelastic fluid as well as for a power law inelastic fluid and by Phan-Thien [7] for a more comprehensive series of constitutive equations both viscoelastic and inelastic. The extensive experimental results presented by Mena *et al.* [5] show beyond doubt that the substantial increases in flow rate which are possible for viscoelastic fluids are mainly due to the shear-rate dependency of the viscosity with a minor influence of the purely elastic effects. This was further corroborated by Phan-Thien [7] and applied to the case of polymer extrusion in oscillatory dies by Casulli *et al.* [8, 9] and later by other authors [10–12].

In this note a simplified analysis is presented in which the influence of elasticity upon the flow is examined. The analysis is concentrated on the cases of oscillating pipe flow and flow between parallel oscillating planes and for a linear viscoelastic constitutive equation. Since the viscosity function is not shear dependent, no increase in flow rate is possible; nevertheless, elastic effects modify the velocity profiles and need to be considered in order to fully explain the experimental increases in flow rate.

2. Mathematical formulation and equation of state. Flow between oscillating parallel plates

We consider the flow between two infinite parallel planes separated a distance $\pm a$ from the z axis of a Cartesian frame of reference whose origin is midway between the planes. The pressure gradient is in the z direction and the oscillations are imposed on the boundaries $y = \pm a$ with a frequency of oscillation n. The physical components of the velocity vector are $u_x = 0$, $u_y = 0$ and $u_z = u(y,t)$ which satisfy the equation of continuity identically. The relevant stress equation of motion is in the z direction and for an incompressible, homogeneous liquid is given by

$$\rho\frac{\partial u}{\partial t} = G + \frac{\partial^2}{\partial y^2}(T'_{yz}) \tag{1}$$

Here ρ is the density of the liquid, G is the generating constant pressure gradient and T'_{yz} is the relevant component of the extra stress tensor. Equation (1) is to be solved subject to the boundary conditions $u = ae^{\text{int}}$ at $y = \pm a$, $\partial u/\partial y = 0$ at $y = 0$ for all t.

We characterize the viscoelastic liquid by equations of state of the form

$$T_{ik} = -pg_{ik} + T'_{ik}, \tag{2}$$

where the extra stress tensor T'_{ik} is given as a function of the relaxation or memory function Ψ as

$$T'_{ik} = 2\int \Psi(t-t')e^{(1)}_{ik}(x,t')\,dt', \tag{3}$$

where

$$\Psi(t-t') = \int \frac{N(\zeta)}{\zeta}e^{-(t-t')/\zeta}\,d\zeta. \tag{4}$$

Following the usual notation T_{ik} is the stress tensor, p an arbitrary isotropic pressure, g_{ik} is the metric tensor of a fixed coordinate system x_i, $e^{(1)}_{ik}$ is the rate of strain tensor and $N(\zeta)$ is the relaxation spectrum.

The liquid represented by Eqs. (2)–(4) falls within the framework of linear viscoelasticity theory and is know as Walters B$'$ liquid [13]. It contains as a special case Oldroyd's B liquid [14] by setting

$$N(\zeta) = \eta_0\frac{\lambda_2}{\lambda_1}\delta(\zeta) + \eta_0\frac{\lambda_1-\lambda_2}{\lambda_1}\delta(\zeta-\lambda_1), \tag{5}$$

where η_0 is the zero shear rate viscosity, λ_1 and λ_2 are the relaxation and retardation times respectively and δ represents a Dirac delta function. Of course if $\lambda_2 = 0$, the Maxwell type constitutive equation is recovered, and if $\lambda_1 = \lambda_2$ the purely viscous case is represented.

For the problem in question, it is easily shown that the relation between T'_{yz} and the velocity component in the direction of flow u, subject to the oscillatory boundary condition at the wall, reduces to

$$T'_{yz} = \frac{\partial u}{\partial y}\int \Psi(x)e^{-inx}\,dx. \tag{6}$$

It may be shown that the velocity gradually becomes a periodic function of time with the same frequency as the velocity of the boundary. We shall consider only this steady periodic state and disregard any transient phenomena.

Equation (1) can be made homogeneous using as dependent variable the departure of the velocity from the steady state value $Ga^2/2\eta_0\left[1-(y/a)^2\right]$, *i.e.*,

$$w(y,t) = u(y,t) - \frac{Ga^2}{2\eta_0}\left[1-\left(\frac{y}{a}\right)^2\right], \tag{7}$$

which yields

$$\frac{\partial w}{\partial t} = \frac{\eta_0}{\rho}\frac{\partial^2 w}{\partial y^2}. \tag{8}$$

with boundary conditions $w(a,t) = \alpha e^{\text{int}}$, $\partial w/\partial y = 0$ at $y = 0$. Here, α is the product of the amplitude times the frequency of the oscillations.

The solution to Eq. (8) is readily found as

$$w(r,t) = \alpha\frac{\cos ky}{\cos ka}e^{\text{int}}. \tag{9}$$

For the general viscoelastic case k is given by

$$k^2 = \frac{-inp}{\eta_0}\int \Psi(\lambda)\exp(in\lambda)\,d\lambda. \tag{10}$$

For the Oldroyd B liquid k reduces to

$$k = \left(\frac{-in\rho}{\eta_0}\right)^{1/2}\left[\frac{1+in\lambda_1}{1+in\lambda_2}\right]^{1/2}, \tag{11}$$

and for the Maxwell fluid:

$$k = \left(\frac{-in\rho}{\eta_0}\right)^{1/2}\left[1+in\lambda\right]^{1/2}. \tag{12}$$

Finally, the Newtonian case is obtained when

$$k = \left(\frac{-in\rho}{\eta_0}\right)^{1/2}. \tag{13}$$

3. Oscillating pipe flow

We refer all physical quantities to cylindrical polar coordinates (r,θ,z), where the z direction is along the axis of the pipe. The pipe has a circular cross section of radius a. The flow is generated by a constant pressure gradient G in the z direction; in addition an oscillatory motion is imposed on the wall $r = a$. This motion is of the form $u = \alpha\exp(e^{\text{int}})$ where n is the frequency of oscillation, the real part is obviously implied and α represents the product of the amplitude and frequency of the oscillations. Again, it may be shown that the velocity gradually becomes a periodic function of time with the same frequency as the velocity of the boundary. We shall consider only this steady periodic state and disregard any transient phenomena. The physical components of the velocity vector under such situation are $u_r = 0$, $u_\theta = 0$ and $u_z = u(r,t)$ which satisfy the equation of continuity identically. The relevant stress equation of motion is in the z direction and for an incompressible, homogeneous liquid is given by

$$\rho\frac{\partial u}{\partial t} = G + \frac{1}{r}\frac{\partial}{\partial r}(rT'_{rz}). \tag{14}$$

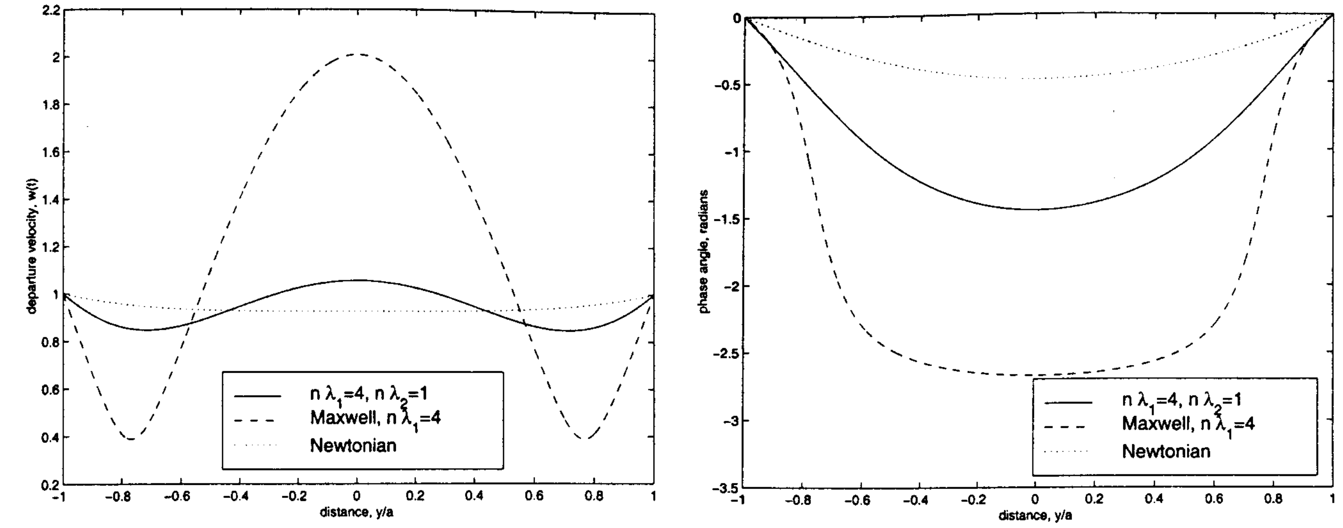
Following the same procedure as in the previous section, we use as dependent variable the departure of the velocity from the steady state value:

$$w(r,t) = u(r,t) - \frac{Ga^2}{4\eta_0}\left[1-\left(\frac{r}{a}\right)^2\right], \tag{15}$$

so that Eq. (14) becomes

$$\frac{\partial w}{\partial t} = \frac{\eta_0}{\rho}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{16}$$

with associated boundary conditions $w(a,t) = \alpha e^{\text{int}}$, $(\partial/\partial r)w(0,t) = 0$.



(a) (b)

FIGURE 1. Parallel planes: (a) velocity departure from the steady state value *vs.* distance and (b) phase angle of velocity departure *vs.* distance between the planes for an Oldroyd fluid with $n\lambda_1 = 4, n\lambda_2 = 1$. The Maxwell model with $n\lambda_2 = 0$ and the Newtonian fluid are shown for comparison.

The solution to Eq. (16) is the found as

$$w(r,t) = \alpha \frac{J_0(kr)}{J_0(ka)} e^{\text{int}}, \quad (17)$$

where J_0 is Bessel's function of order zero of the first kind. With the use of Eq. (15) one readily obtains

$$u(r,t) = \alpha \frac{J_0(kr)}{J_0(ka)} e^{\text{int}} + \frac{Ga^2}{4\eta_0}(1 - R^2), \quad (18)$$

where $R = r/a$. The various values of k being the same as in the preceeding section.

4. Results and conclusions

Before computing any results from the above solutions, it is useful to examine the behavior of the functions in Eqs. (9) and (17). For the sake of simplicity we choose to examine the solution for flow between parallel plates; the solution for pipe flow is expected to have the same general form although slightly more complicated in nature. If one examines the quantity of interest in Eq. (9), *i.e.*,

$$W = \frac{\cos kay^*}{\cos ka}, \quad (19)$$

where we have set $y^* = y/a$ and y^* varies between 0 and 1, the product ka is complex and may be expressed as

$$ka = \beta - i\gamma. \quad (20)$$

Restrictions are imposed on β and γ since n, ρ, η_0 and λ are all real and positive; then β and γ are both real and positive; therefore, it follows that:

(i) since $\gamma > 0$ the modulus W is never zero or infinite;

(ii) if $\beta \leq \gamma$, W increases monotonically to the value unity as y^* goes from 0 to 1. Note that $\beta = \gamma$ is the Newtonian case.

The limiting cases may also be examined in the context of linear viscoelasticity theory; for example the complex modulus η^* is

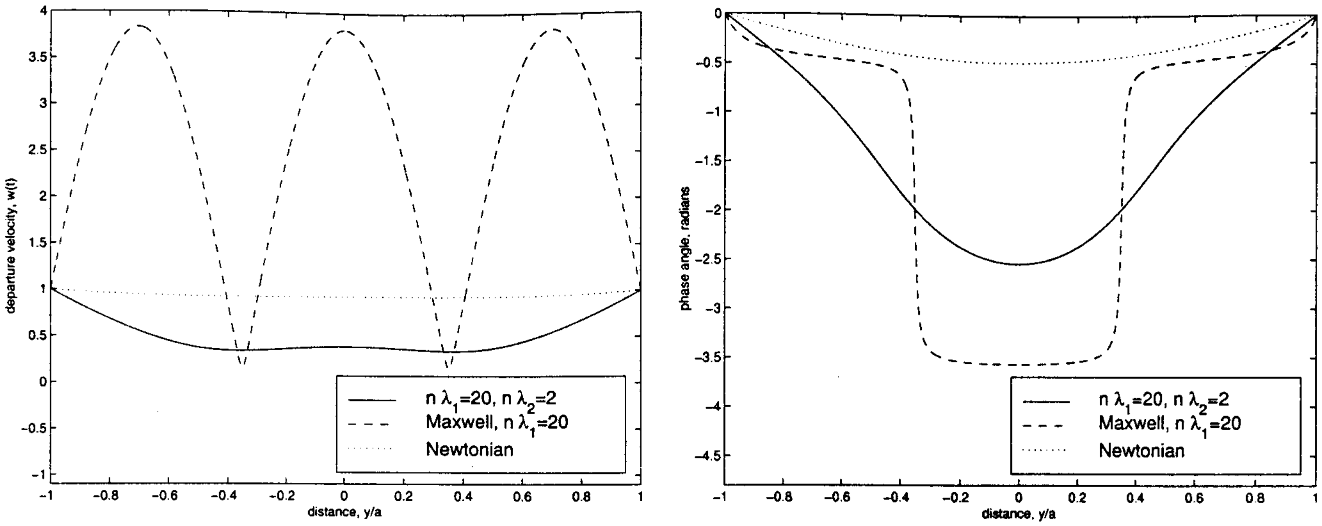
$$\frac{\eta^*}{\eta_0} = \frac{1 + in\lambda_2}{1 + in\lambda_1}, \quad (21)$$

therefore Eq. (20) may then be expressed as

$$ka = (a^2 n\rho)^{1/2} \left(\frac{-i}{\eta^*}\right)^{1/2}, \quad (22)$$

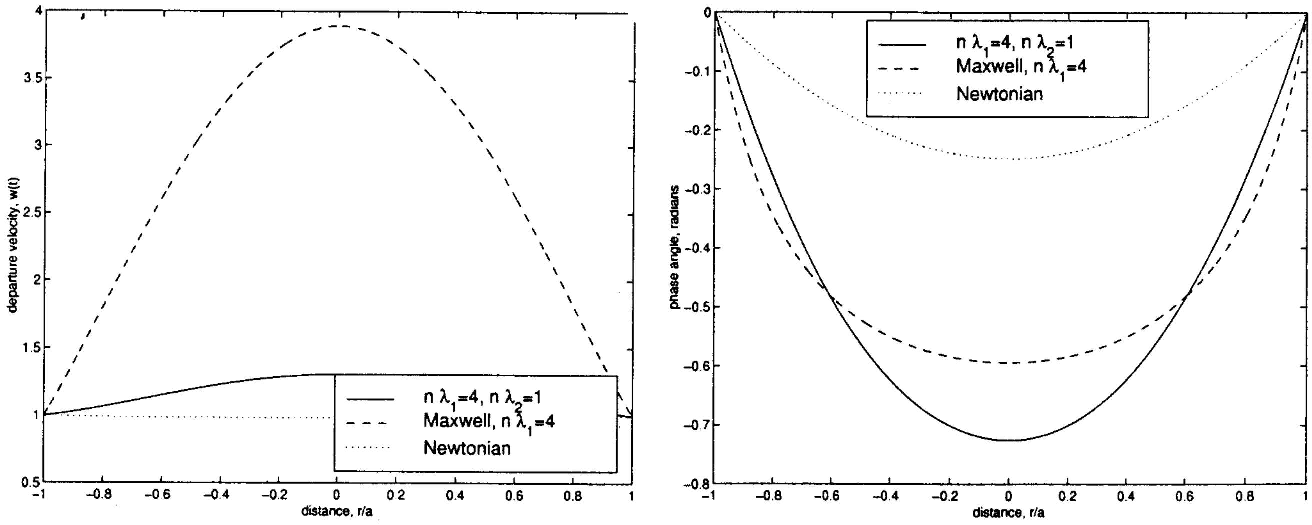
while Eq. (21) can be written as $\eta^* = \eta' - i\eta''$ where $\eta', \eta'' > 0$.

If the material is an elastic solid we have $\eta'' = 0$ and thus $\gamma = 0$. Hence W becomes infinite as expected except for certain nodes. The results are shown in Figs. 1 and 2, where comparison is made between the oscillating velocity profiles for a Newtonian, Maxwell and Oldroyd fluids. The velocity departure and the corresponding phase angle has been plotted for different values of $n\lambda_1$ and $n\lambda_2$ for the Oldroyd fluid with the Maxwell and Newtonian fluids as special cases. It may be observed that as the relaxation time or as the frequency is increased, the Maxwell model shows unrealistic behavior, such as resonance effects that have misled some authors [15] into assuming that an increase in flow rate is possible. It is interesting to point out that the apparent resonance behavior appearing in solutions to unsteady flow problems when a Maxwell type constitutive equation is used [15], is a well known problem inherent to the nature of the approximation of that particular equation (see for example, Tanner [16]). This



(a) (b)

FIGURE 2. Parallel planes: velocity departure from the steady state value *vs.* distance and (b) phase angle of velocity departure *vs.* distance between the planes for an Oldroyd fluid with $n\lambda_1 = 20, n\lambda_2 = 2$. The Maxwell model with $n\lambda_2 = 0$ and the Newtonian fluid are shown for comparison.



(a) (b)

FIGURE 3. Pipe flow: velocity departure from the steady state Poiseuille value *vs.* radial distance and (b) phase angle of velocity departure *vs.* radial distance for an Oldroyd fluid with $n\lambda_1 = 4, n\lambda_2 = 1$. The Maxwell model with $n\lambda_2 = 0$ and the Newtonian fluid are shown for comparison.

"resonance effect" disappears as soon as the relaxation spectrum of the fluid is represented by more than one relaxation time (as is the case for any real viscoelastic fluid); this is clearly demonstrated by the simple analysis of the oscillating flow between parallel plates presented above.

On the other hand, it is readily observed that the integration of this velocity departure from the steady Poiseuille flow over the cross sectional area is zero since the function is periodic. Therefore, the contribution to the total flow rate is non-existent.

Figures 3 and 4 show similar results for the case of oscillating pipe flow. Finally, Figs. 5 and 6 show the velocity profiles during a period of oscillation. It may be observed that for the same parameters, the influence of elasticity upon the velocity is more noticeable in the case of flow between parallel plates than in the case of oscillatory pipe flow.

5. Final remarks

It should be made clear that regardless of whether the flow is pulsatile (oscillatory pressure gradient) or in the case of oscillating walls, in the frame of linear viscoelasticity the resulting flow rate must remain unchanged from its Newtonian

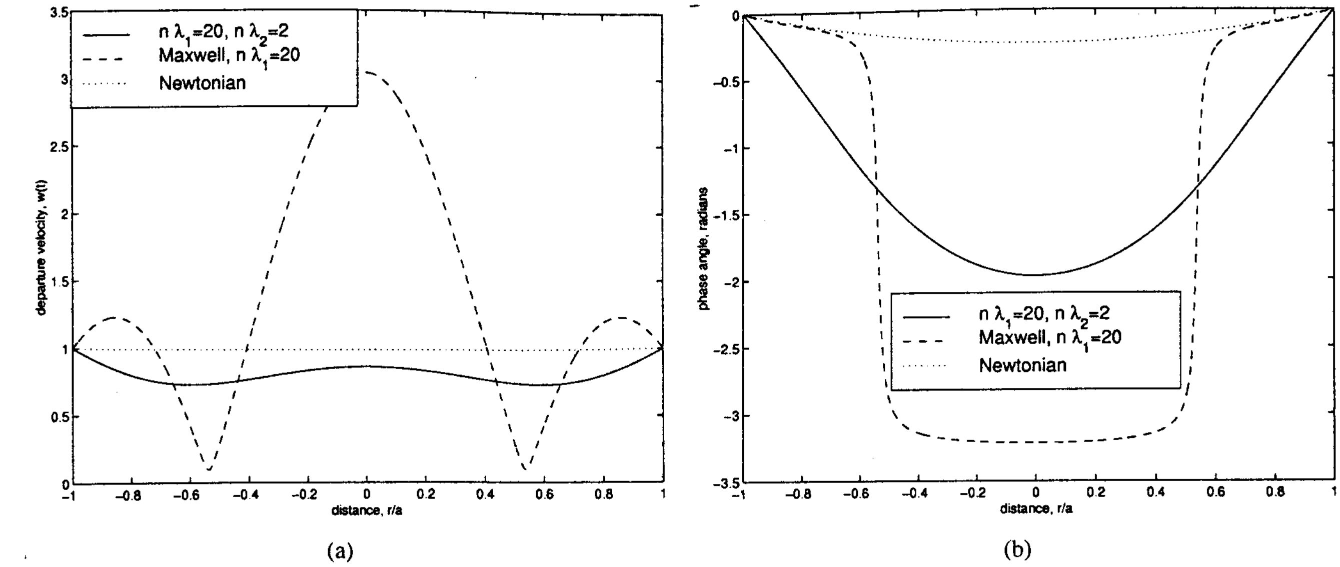


FIGURE 4. Pipe flow: velocity departure from the steady state value *vs.* radial distance and (b) phase angle of velocity departure *vs.* radial distance for an Oldroyd fluid with $n\lambda_1 = 20, n\lambda_2 = 2$. The Maxwell model with $n\lambda_2 = 0$ and the Newtonian fluid are shown for comparison.

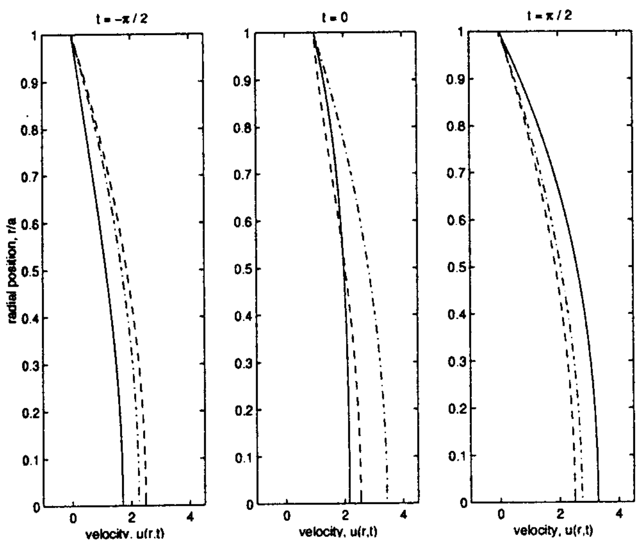


FIGURE 5. Pipe flow: velocity profiles at various times. The z axis is located at the center of the pipe. Oldroyd fluid with $n\lambda_1 = 20, n\lambda_2 = 2$ — — — — —; Maxwell model ———; Newtonian -·-·-·-.

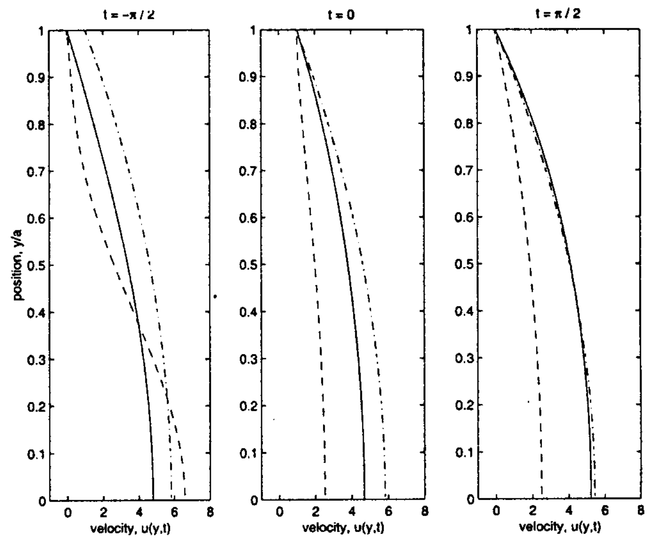


FIGURE 6. Parallel planes: velocity profiles at various times. The z axis is located midway between the planes. Oldroyd fluid with $n\lambda_1 = 20, n\lambda_2 = 2$ — — — — —; Maxwell model ———; Newtonian -·-·-·-.

value. The only stresses that are relevant are the viscous stresses at the walls. If the fluid is linear (as in the case of the Maxwell model) the effects of elasticity cannot alter the viscous stress and therefore the flow rate must remain unchanged from the purely viscous case. For the flow rate to be altered, the viscosity of the fluid must be a allowed to change as a function of the shear rate; this was demonstrated experimentally in Ref. 5 and discussed at length by Phan-Thien in Ref. 7 who solved the problem for a number of constitutive equations with variable viscosity. If the flow rate is caused by a pressure gradient whether constant or pulsating over a mean value, the main flow is a steady Poiseuille flow; if oscillations are superposed and the fluid is within the frame of linear viscoelasticity, the oscillations may alter the instantaneos velocity profile due to elastic effects but they cannot change the mean flow rate.

* Corresponding author.

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