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On the time estimate for start-up of pipe flows in a Bingham fluid — a proof of the result due to Glowinski, Lions and Trèmoliéres

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Abstract

A proof of the result due to Glowinski, Lions and Trèmoliéres on the time estimate for start-up of pipe flows in a Bingham fluid is presented. This result states that the norm of the unsteady flow approaches that of the steady flow exponentially, with the density and viscosity of the fluid, and the lowest eigenvalue of the Laplacian over the cross-section of the pipe playing significant roles.

Keywords: Bingham fluid; Pipe flow; Variational inequality; Time estimate

1. Introduction

Suppose that $c > 0$ is a constant pressure drop per unit length, which is sufficient to overcome the yield stress in a Bingham fluid. Further, under this pressure drop, let the fluid attain a steady velocity field $u_\infty = u_\infty(x, y)$ in a pipe of arbitrary cross-section, defined by $\mathcal{A}$ in the x–y plane. Further, let us assume that for every initial value $u_0 = u_0(x, y)$, the unsteady pipe flow with the same, constant pressure drop $c > 0$, has a unique solution $u(t) = u(x, y, t)$. By $\|\cdot\|$, let us denote the L_2 norm of a function over $\mathcal{A}$, i.e.

$$\|f\|^2 = \int_{\mathcal{A}} f^2 \,\mathrm{d}a. \tag{1.1}$$

In a start-up problem, we are interested in how the norm of the difference $\|u_\infty - u(t)\| \to 0$ as $t \to \infty$. That is, if $w(t) = u_\infty - u(t)$ is the unique solution to the time-dependent flow problem with an initial

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value $w(0) = u_\infty - u(0)$, the problem that has to be investigated is the rate of decay of $\|w(t)\|$ as $t \to \infty$. If the fluid were Newtonian, the velocity field $w(t)$ occurs under a zero pressure drop, and it can be shown that $w(t) \to 0$ as $t \to \infty$; for example, see [1]. Note that this is a result concerning the pointwise convergence of $w(t)$ to zero from which the norm limit of zero is clearly established.

Here, the equations governing the flow of the yield stress fluid are non-linear, and hence $w(t)$ may not occur under a zero pressure drop. Instead of finding this pressure term, which may be unsteady, it will be proved here that

$$\rho\left(\frac{\partial(u_\infty - u(t))}{\partial t}, u_\infty - u(t)\right) + \eta a(u_\infty - u(t), u_\infty - u(t)) \leq 0, \tag{1.2}$$

where ρ is the density, and η is the constant viscosity of the fluid.

In Eq. (1.2), the dot product $(\cdot,\cdot)$ and the dissipation term $a(\cdot,\cdot)$ for the Bingham fluid appear. They are defined through

$$(f, g) = \int_{\mathcal{A}} fg\,\mathrm{d}a, \qquad a(u, v) = \int_{\mathcal{A}} \nabla u \cdot \nabla v\,\mathrm{d}a, \tag{1.3}$$

where f and g are any smooth functions, u and v are any two smooth velocity fields, which vanish on the boundary of the pipe, and ∇ is the two dimensional gradient operator. Now, it is known that the dissipation term obeys the coercive inequality

$$a(u_\infty - u(t), u_\infty - u(t)) \geq \lambda_1 \|u_\infty - u(t)\|^2, \tag{1.4}$$

where $\lambda_1 > 0$ is the least eigenvalue of the Laplace operator in the domain $\mathcal{A}$. That is, $\lambda_1 > 0$ is the minimum eigenvalue of the boundary value problem

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \lambda w = 0, \tag{1.5}$$

with $w = 0$ on the boundary. The proof of the inequality (1.4) is not given here, for it is nothing but a statement about the variational characterisation of the least eigenvalue of the Laplace operator (for example, see Theorem 4.2 in [2]).

Since,

$$\left(\frac{\partial(u_\infty - u(t))}{\partial t}, u_\infty - u(t)\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_\infty - u(t)\|^2, \tag{1.6}$$

the inequality (1.2) now becomes

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\|u_\infty - u(t)\| + \eta_0\lambda_1\|u_\infty - u(t)\| \leq 0. \tag{1.7}$$

This proves that

$$\|u_\infty - u(t)\| \leq \|u_\infty - u(0)\|\,\mathrm{e}^{-\eta_0\lambda_1 t/\rho}, \tag{1.8}$$

which shows the way the norm $\|w(t)\|$ of the difference $w(t)$ approaches zero as $t \to \infty$. The above result is not new, for it appears as Remark 3.1 in Appendix 6 of the book by Glowinski et al. [3], or as Exercise 6.1 in Chapter III of the book by Glowinski [4], with the reader being asked to establish it.

However, no proof appears in these two treatises, and we are not aware of any published version of a proof. We believe that it may be of interest to see a detailed argument, especially if one is interested in expanding the result to more general yield stress fluids, such as Herschel–Bulkley fluids. Thus, the goal of the present note is to prove the inequality in Eq. (1.2).

Before proceeding with the mathematical details, we wish to point out that it is the viscosity which determines the rate of growth of the transient solution to the final value, while the yield stress puts a lower bound on the pressure drop for the flow to persist. Indeed, Glowinski et al. in Theorem 3.2, Appendix 6 of [3] and Glowinski in Theorem 6.2, Chapter III of [4] prove that if $c < \tau_y \beta$, then the pipe flow will be non-zero for a finite time only, where

$$\beta = \min_{v \neq 0} \frac{\int_{\mathcal{A}} |\nabla v| \, \mathrm{d}a}{\|v\|} > 0. \tag{1.9}$$

We do not use this result here, except to assume that $c \geq \tau_y \beta$ for the unsteady pipe flow $u(t)$ to grow and approach its steady state counterpart u_∞.

2. The constitutive relations

Let $\boldsymbol{D}(\boldsymbol{v})$ be the rate of deformation tensor derived from a velocity field $\boldsymbol{v}$ which is isochoric, i.e. $\nabla \cdot \boldsymbol{v} = 0$. Now,

$$2D_{ij}(\boldsymbol{v}) = v_{i,j} + v_{j,i}. \tag{2.1}$$

Since $\boldsymbol{D}$ is symmetric, the second invariant K of $\boldsymbol{D}$ is given by

$$K^2(\boldsymbol{v}) = D_{ij}(\boldsymbol{v}) D_{ij}(\boldsymbol{v}). \tag{2.2}$$

Let the stress tensor $\boldsymbol{S}$ in an incompressible yield stress fluid be written as $S_{ij} = -p\delta_{ij} + \tau_{ij}$, where τ_{ij} is the extra stress tensor. The equations defining the Bingham fluid are

$$D_{ij} = 0, \qquad T \leq \sqrt{2}\tau_y, \tag{2.3}$$

$$\tau_{ij} = 2\eta D_{ij} + \frac{\sqrt{2}\tau_y}{K} D_{ij}, \qquad T > \sqrt{2}\tau_y, \tag{2.4}$$

where η and τ_y are, respectively, the constant viscosity and the constant yield stress of the fluid. The second invariant T of the extra stress tensor is

$$T^2(\tau) = \tau_{ij}\tau_{ij}. \tag{2.5}$$

The physical meaning of Eq. (2.3) is that the fluid moves as a rigid body or is at rest over all points where the inequality holds; that of Eq. (2.4), is that the fluid yields and deforms, and an explicit constitutive relation applies when $\boldsymbol{D} \neq \boldsymbol{0}$.

3. Start-up problem for a Bingham fluid

Consider the unsteady flow in a pipe where all velocity fields, such as $\boldsymbol{u}$ and $\boldsymbol{v}$, have a non-zero component in the axial direction only. That is, $\boldsymbol{u} = u(x, y, t)\boldsymbol{k}$, and so on. In this case, it is easily shown that the shear stresses τ_{xz} and τ_{yz} depend on (x, y, t) only, while the normal stress $\tau_{zz} = 0$. The pressure p depends on z only, i.e. $p = -cz$, where $c > 0$ is the constant pressure drop per unit length. We shall now state the variational inequality applicable to the pipe flow, which leads to Eq. (1.7). For a derivation of the inequality, see [3–7].

For the assumed velocity fields,

$$D_{ij}(\boldsymbol{u})D_{ij}(\boldsymbol{v}-\boldsymbol{u}) = \tfrac{1}{2}\nabla u \cdot \nabla(v-u), \tag{3.1}$$

and

$$K^2(u) = \tfrac{1}{2}\nabla u \cdot \nabla u, \tag{3.2}$$

where ∇ is the two-dimensional gradient operator.

Now, set the body force $\boldsymbol{b} = \boldsymbol{0}$, or absorb it into the pressure term, if the body force is that due to gravity. Also, note that the acceleration vector is given by

$$\boldsymbol{a} = \frac{\partial u}{\partial t}\boldsymbol{k}. \tag{3.3}$$

Let us now define four integrals which are necessary to explain the terms used in the variational inequality. They are

$$a(u, v-u) = \int_{\mathcal{A}} \nabla u \cdot \nabla(v-u)\, \mathrm{d}a, \tag{3.4}$$

$$j(v) = \int_{\mathcal{A}} |\nabla v|\, \mathrm{d}a, \tag{3.5}$$

$$(c, v-u) = \int_{\mathcal{A}} c(v-u)\, \mathrm{d}a, \tag{3.6}$$

$$\left(\frac{\partial u}{\partial t}, v-u\right) = \int_{\mathcal{A}} \frac{\partial u}{\partial t}(v-u)\, \mathrm{d}a. \tag{3.7}$$

It is shown in [3–7] that one is led to following the variational inequality for unsteady pipe flows

$$\rho\left(\frac{\partial u}{\partial t}, v-u\right) + \eta a(u, v-u) + \tau_y j(v) - \tau_y j(u) \geq (c, v-u), \tag{3.8}$$

where u is the solution to the pipe flow problem under the constant pressure gradient, and v is any trial velocity field which obeys the same boundary condition as u. The crucial point in exploiting the above inequality lies in the fact that although u is unsteady, v may be steady or unsteady.

If the solution u_∞ is steady, the corresponding inequality is

$$\eta a(u_\infty, v-u_\infty) + \tau_y j(v) - \tau_y j(u_\infty) \geq (c, v-u_\infty). \tag{3.9}$$

Again, v may be steady or unsteady.

We can derive the exact form of the equation satisfied by the true solution field u from Eq. (3.8). Assume that the trial velocity field $v = 2u$. Note that this is permissible, because $u = 0$ on the boundary of the pipe means that the trial velocity field $v = 2u$ satisfies this boundary condition. Then, Eq. (3.8) is converted to

$$\rho\left(\frac{\partial u}{\partial t}, u\right) + \eta a(u, u) + \tau_y j(u) \geq (c, u). \tag{3.10}$$

If the trial velocity field $v = 0$, we obtain

$$-\rho\left(\frac{\partial u}{\partial t}, u\right) - \eta a(u, u) - \tau_y j(u) \geq -(c, u). \tag{3.11}$$

Comparing Eqs. (3.10) and (3.11), it can be seen that we have an equality:

$$\rho\left(\frac{\partial u}{\partial t}, u\right) + \eta a(u, u) + \tau_y j(u) = (c, u). \tag{3.12}$$

If the solution is steady, say u_∞, the equation satisfied by it is

$$\eta a(u_\infty, u_\infty) + \tau_y j(u_\infty) = (c, u_\infty). \tag{3.13}$$

In the next section, we shall exploit Eqs. (3.8), (3.9), (3.12) and (3.13) repeatedly to derive the result announced in Eq. (1.7) above. The proof is simple and straightforward.

4. The proof

First of all, we note from Eq. (3.4) that $a(\cdot, \cdot)$ is bilinear and symmetric, i.e.

$$a(u(t), u_\infty - u(t)) = a(u(t), u_\infty) - a(u(t), u(t)), \qquad a(u(t), u_\infty) = a(u_\infty, u(t)). \tag{4.1}$$

As well, from Eqs. (3.12) and (3.13), it follows that

$$(c, u_\infty - u(t)) = \eta a(u_\infty, u_\infty) + \tau_y j(u_\infty) - \rho\left(\frac{\partial u}{\partial t}, u\right) - \eta a(u, u) - \tau_y j(u). \tag{4.2}$$

Next, if we put $v = u_\infty$ in Eq. (3.8), we find that

$$\rho\left(\frac{\partial u(t)}{\partial t}, u_\infty - u(t)\right) + \eta a(u(t), u_\infty - u(t)) + \tau_y j(u_\infty) - \tau_y j(u(t)) \geq (c, u_\infty - u(t)). \tag{4.3}$$

Using Eqs. (4.2) and (4.3), it is easily shown that

$$\rho\left(\frac{\partial u(t)}{\partial t}, u_\infty\right) + \eta a(u(t), u_\infty) \geq \eta a(u_\infty, u_\infty). \tag{4.4}$$

This result shows the effect of the inner product term, involving inertia, and the hybrid dissipation term on the true dissipation term in a steady flow.

Finally, if we put $v = u(t)$ in Eq. (3.9), we obtain

$$\eta a(u_\infty, u(t) - u_\infty) + \tau_y j(u(t)) - \tau_y j(u_\infty) \geq (c, u(t) - u_\infty). \tag{4.5}$$

Now, using Eqs. (4.1) and (4.2) in Eq. (4.5), we find that

$$\eta a(u(t), u_\infty) \geq \rho\left(\frac{\partial u(t)}{\partial t}, u(t)\right) + \eta a(u(t), u(t)). \tag{4.6}$$

Combining Eqs. (4.4) and (4.6), it is seen that

$$\rho\left(\frac{\partial u(t)}{\partial t}, u_\infty\right) + 2\eta a(u(t), u_\infty) \geq \rho\left(\frac{\partial u(t)}{\partial t}, u(t)\right) + \eta a(u(t), u(t)) + \eta a(u_\infty, u_\infty). \tag{4.7}$$

The above results can now be put to use. We begin with

$$\left(\frac{\partial(u_\infty - u(t))}{\partial t}, u_\infty - u(t)\right) = -\left(\frac{\partial u(t)}{\partial t}, u_\infty\right) + \left(\frac{\partial u(t)}{\partial t}, u(t)\right), \tag{4.8}$$

since u_∞ is steady. Next, from Eq. (4.1), we obtain

$$a(u_\infty - u(t), u_\infty - u(t)) = a(u_\infty, u_\infty) + a(u(t), u(t)) - 2a(u(t), u_\infty). \tag{4.9}$$

Utilising Eqs. (4.7), (4.8) and (4.9), it is seen that

$$\rho\left(\frac{\partial(u_\infty - u(t))}{\partial t}, u_\infty - u(t)\right) + \eta a(u_\infty - u(t), u_\infty - u(t)) \leq 0, \tag{4.10}$$

which is Eq. (1.2), and the proof is complete.

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