INVESTIGACIÓN REVISTA MEXICANA DE FÍSICA 47 (4) 351–356 AGOSTO 2001

Viscous dissipation of a power law fluid in an oscillatory pipe flow

J.R. Herrera-Velarde
Instituto Tecnológico de Zacatepec
Av. Instituto Tecnológico 27, 67980 Zacatepec, Mor., Mexico
R. Zenit and B. Mena*
Instituto de Investigación en Materiales, Universidad Nacional Autónoma de México
Apartado postal 70-360, Cd. Universitaria, 04510 Coyoacán, México, D.F., Mexico

Recibido el 16 de febrero de 2001; aceptado el 11 de mayo de 2001

The flow field in an oscillatory pipe is studied theoretically for a generalized newtonian fluid model. The velocity and temperature fields are obtained for the case in which the mean velocity caused by the pressure gradient is of the same order as the oscillation velocity. The momentum and the energy conservation equations are solved and analytic expressions for the velocity and temperature fields are found. The nature of the velocity and temperature profiles is explored for a range of parameters. In general, it can be concluded that the temperature rise within the fluid increases with the speed of oscillation as the value of the power parameter increases. An effective heat transfer coefficient is calculated and plotted as a function of the normalized oscillation speed. The cases of a newtonian, shear-thinning and shear-thickening fluid are analyzed.

Keywords: Viscous dissipation; power law fluid; oscillatory flow

Se presenta un estudio teórico de un flujo oscilatorio en una tubería para un fluido tipo ley de potencia. Las ecuaciones de momentum y energía se resuelven y se encuentran soluciones analíticas para los campos de velocidad y temperatura. Se obtienen resultados para el caso en que la velocidad media debido al gradiente de presión es de la misma magnitud que la velocidad de oscilación. Se exploran la naturaleza de los campos de velocidad y temperatura como función de los parámetros dominantes. En general, concluimos que el incremento de temperatura en el fluido aumenta como función de la rapidez de oscilación y del parámetro de potencia. Se calcula un coeficiente de transferencia efectivo y se grafica como función de la rapidez de oscilación adimensional. Los líquidos newtoniano, pseudo plástico y dilatante son analizados.

Descriptores: Disipación viscosa; ley de potencia; flujo oscilatorio

PACS: 47.50.+d; 47.60.+i

1. Introduction

The majority of studies that involve the flow of a viscous fluid assume an isothermal state; however, in practice, many flows are far from this situation. The combination of high viscosities and large velocity gradients may result in a significant increase of the fluid temperature resulting from viscous dissipation. This effect is used, for example, in extrusion processes where the temperature increase is used to accelerate the melting of the material.

The heat transfer in ducts has been extensively studied for the case of newtonian fluids. Greatz [?] solved the classic problem of forced heat convection in a pipe subjected to different boundary conditions, neglecting axial conduction. More recently, Yin [?] solved Greatz problem analytically taking into account the axial conduction in the fluid as well as in the pipe, concluding that the axial conduction indeed plays an important role in the entrance region. Much less work has been devoted to study heat transfer problems involving non-newtonian viscous fluids. It is known, however, that the non-newtonian properties of a fluid can significantly change the heat transfer characteristics [?]. The coupling between the equations of motion and the energy equation can be achieved through the material properties of the fluid assumed, as a first approximation, as temperature-independent constitutive relations.

The flow of polymer solutions and melts in oscillating pipes has been studied extensively by Mena and coworkers [?, ?, ?, ?]. Imposing longitudinal oscillations on a viscoelastic fluid, the velocity fields and the pressure drop change were studied, but only considering the isothermal case. Among many other conclusions, these authors state that the oscillation of an exit die of an extrusion process alters the mechanical properties of the extruded product and causes an important decrease of the pressure drop across the die. More recently, Herrera-Velarde [?] studied theoretically the heat dissipation on oscillatory flows considering a viscoelastic model.

In order to understand the effect of oscillations in a polymer extrusion process, the non iso-thermal case must be studied. This paper presents the study of the pipe flow in which the pipe oscillates in the main direction of the flow. Results are obtained for an inelastic power law fluid. The temperature profiles are obtained using the velocity profiles and by assuming temperature independent properties. The coefficient of heat transfer is obtained and presented for a range of flow conditions and fluid properties.

2. Problem definition

Consider a fluid that flows in a duct with circular cross-section of radius r resulting from uniform pressure gradient ∇P

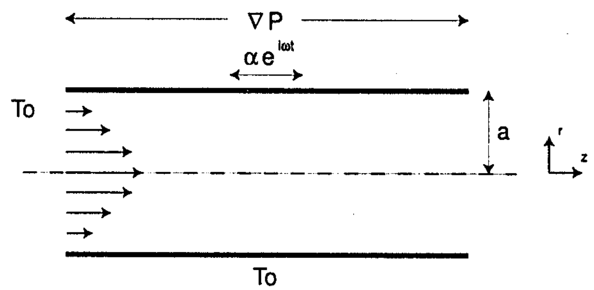

FIGURE 1. Schematic of the flow entering the oscillating die with a developed velocity profile, a mean temperature T_0 and subjected to an oscillating wall with constant temperature T_0.

in the axial direction of the pipe, z. Additionally, the duct oscillates in the direction parallel to the flow imposing an additional shear stress to the fluid. A schematic diagram of the flow is shown in Fig. 1.

Considering the conservation equations for an incompressible and uniform liquid,

$$\nabla \cdot \vec{u} = 0, \quad (1)$$

$$\rho\left(\frac{\partial \vec{u}}{\partial t}\right) = -\nabla P + \nabla \cdot \tilde{\tau}, \quad (2)$$

$$\rho C_p(\vec{u} \cdot \nabla T) + \nabla \cdot (k \nabla T) = \tilde{\tau} : \nabla \vec{u}, \quad (3)$$

where $\vec{u}$ is the velocity vector, P is the scalar pressure, $\tilde{\tau}$ is the deviatoric stress tensor, ρ is the density, C_p is the specific heat, k is the thermal conductivity and T is the temperature.

For the geometry of the flow showed in Fig. 1, the velocity vector reduces to $(u_r, u_\theta, u_z) = (0, 0, u_z)$ which satisfies the equation of continuity identically [Eq. (1)]. The momentum equation [Eq. (2)] in the z-direction is reduced to

$$\rho \frac{\partial u_z}{\partial t} = G + \frac{1}{r}\frac{\partial}{\partial r}(r \tau_{rz}), \quad (4)$$

where $G = -\partial P / \partial z$ is the constant pressure gradient and τ_{rz} is the component of the stress tensor in the axial direction of the pipe. The boundary conditions for this equation are

1) $u_z = \alpha \mathrm{Re}[\exp(i\omega t)]$ at $r = a$.

2) $\partial u_z / \partial r = 0$ at $r = 0$.

The energy equation [Eq. (3)] for a fully developed temperature field for the same flow, neglecting the axial conduction, reduces to

$$\rho C_p u_z \left(\frac{\partial T}{\partial z}\right) = k \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \tau_{rz}\left(\frac{\partial u_z}{\partial r}\right). \quad (5)$$

We consider that the temperature profile of the fluid is constant with T_0 at the entrance of the oscillating wall. The oscillating wall temperature is also held constant at T_0. Hence, the boundary conditions are given by,

3) $T = T_0$ at $r = a$ for all z.

4) $\partial T / \partial r = 0$ at $r = 0$ for all z.

The above set of equations can be solved when a model for the rheological behavior is given. To obtain analytical solutions, a generalized newtonian model is considered.

2.1. Generalized newtonian model

The generalized newtonian model considers a modified viscosity-shear rate relationship to account for the case in which the viscosity of the fluid depends on the shear rate. A well-known generalized model is the "power law" model [?], in which the viscosity of the fluid depends on the magnitude of the strain rate,

$$\eta = m\dot{e}^{\,n-1}, \quad (6)$$

where n and m are constants characteristics of a particular fluid. The magnitude of the strain rate is defined as $\dot{e} = \sqrt{\sum_i \sum_j e_{ij} e_{ij}/2}$. Hence, the ij component of the extra stress tensor is

$$\tilde{\tau} = m e_{ij}^{n}. \quad (7)$$

The relevant component of the stress tensor for the problem of interest is τ_{rz}, which can be expressed simply by

$$\tau_{rz} = m\left(\frac{\partial u_z}{\partial r}\right)^n. \quad (8)$$

Note that if $n = 1$ the newtonian case is recovered where the value of m corresponds to the shear viscosity η.

3. Results

For the rheological model discussed above, the velocity and temperature field are obtained. We consider the case in which the mean velocity caused by the pressure gradient is of the same order as the oscillating velocity. For simplicity, the material parameters (m, ρ, k, C_p) are all considered to be of order 1.

To solve the momentum conservation [Eq. (4)] we assume that the axial velocity component u_z can be decomposed in $u_0 = u_0(t)$, the motion due to the periodic oscillation of the walls, and u_p resulting from the pressure gradient,

$$u_z = u_0 + u_p, \quad (9)$$

where ω and A are the frequency and amplitude of the oscillation. If we take $u_0 = \Re\{\omega A \exp(i\omega t)\}$, Eq. (4) can be expressed as

$$i\omega^2 \rho A \exp(i\omega t) = G + \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}). \quad (10)$$

If the pressure gradient G is constant, the expression above can be integrated with respect to r to obtain the general solution for the component τ_{rz} of the stress tensor. With the condition that the stress must be finite at $r = 0$, we obtain

$$\tau_{rz} = \frac{r}{2}\left[-G + \rho i\omega^2 A \exp(i\omega t)\right]. \quad (11)$$

Using this expression in Eq. 8, we obtain

$$\left(\frac{du_z}{dr}\right) = \left(\frac{r}{2m}\right)^{\frac{1}{n}} \left[- G + i\omega^2 \rho A \exp(i\omega t)\right]^{\frac{1}{n}}, \quad (12)$$

which can be integrated with respect to r to obtain the velocity profile using the wall velocity as a boundary condition, $u_z(r = a) = \alpha \exp(i\omega t)$.

We must note that the parameter G is indeed a negative number $(-\partial P/\partial z)$ that determines the mean direction of the flow (form left to right in this case). We can simplify the mathematical analysis if we consider that

$$-G + i\omega^2\alpha \exp(i\omega t) = -\left[|G| + i\omega^2\alpha\exp(i\omega t)\right],$$

to avoid the case in which a negative number is raised to a fractional power. Adopting this sign convention does not affect the nature of the solution and simplifies the analysis significantly. Hence,

$$u_z = \left(\frac{n}{1+n}\right)\left(\frac{|G| + i\omega\alpha\rho\exp(i\omega t)}{2m}\right)^{\frac{1}{n}} \times \left(a^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}\right) + \alpha\exp(i\omega t). \quad (13)$$

A non-dimensional velocity is defined as

$$\frac{u_z}{u_m} = \left(\frac{|G| + i\omega\alpha\exp(i\omega t)^{\frac{1}{n}}}{|G|}\right)^{\frac{1}{n}}\left[1 - \left(\frac{r}{a}\right)^{\frac{1}{n}+1}\right] + \frac{\alpha\exp(i\omega t)}{\left(\frac{n}{1+n}\right)\left(\frac{|G|}{2m}\right)^{\frac{1}{n}} a^{\frac{1}{n}+1}} \quad (14)$$

where u_m is the maximum velocity for the non-oscillating Poiseuille pipe flow given by

$$u_m = \left(\frac{n}{1+n}\right)\left(\frac{|G|}{2m}\right)^{\frac{1}{n}} a^{\frac{1}{n}+1}.$$

To make the magnitude of the oscillation velocity be of the same order as that of the Poiseuille flow, the pressure gradient G is calculated in terms of the oscillation speed

$$\alpha \approx u_m,$$

hence

$$|G| = 2m\left(\frac{\alpha(1+n)}{n a^{\frac{1}{n}+1}}\right)^n.$$

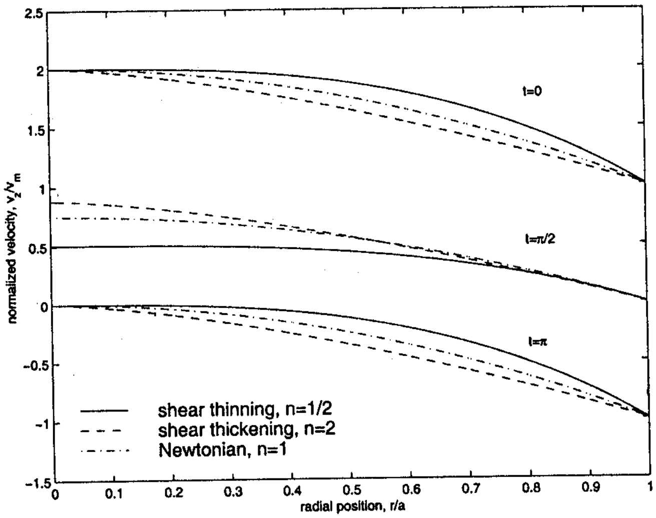

FIGURE 2. Dimensionless velocity as a function of dimensionless position or power law fluids with: a) (———) shear thinning liquid, $n = 1/2$, b) (– – – –) shear thickening liquid, $n = 2$, and c) (– · – · –) newtonian liquid, $n = 1$. For $\omega\rho/\eta_0 = 1$, $\alpha/u_m = 1$, and $m = 1$.

Figure 2 shows the non-dimensional velocity profiles for three cases of fluids: a shear thinning fluid ($n < 1$), a shear thickening fluid ($n > 1$) and a newtonian fluid ($n = 1$).

The energy equation [Eq. (5)] can be written for a power law fluid leading to

$$\rho C_p u_z \frac{\Delta T}{\Delta z} = k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + m\left(\frac{du_z}{dr}\right)^{n+1}, \quad (15)$$

where $\Delta T/\Delta z$ is considered to be a constant parameter. The above equation can be integrated for the given set of boundary conditions since the velocity profile [Eq. (13)] is known,

$$T = T_0 + C_1\left\{\int\frac{1}{r}\left(\int r' u_z(r')\,dr'\right)dr - \left[\int\frac{1}{r}\left(\int r' u_z(r')\,dr'\right)dr\right]_{r=a}\right\}$$
$$-C_2\left\{\int\frac{1}{r}\left(\int r'\left(\frac{\partial u_z(r')}{\partial r'}\right)^{n+1}dr'\right)dr - \left[\int\frac{1}{r}\left(\int r'\left(\frac{\partial u_z(r')}{\partial r'}\right)^{n+1}dr'\right)dr\right]_{r=a}\right\}, \quad (16)$$

where

$$C_1 = \frac{\rho C_p}{k}\left(\frac{\Delta T}{\Delta z}\right) \quad \text{and} \quad C_2 = \frac{m}{k}.$$

An analytic solution can be found for the above expression resulting in

$$T - T_0 = C_1\left[\frac{C_3 a^{\frac{1}{n}+1} + C_4}{4}(r^2 - a^2) - \frac{C_3 n^2}{(1+3n)^2}\left(r^{\frac{1}{n}+3} - a^{\frac{1}{n}+3}\right)\right] - \frac{C_2 n^2}{(1+3n)^2}\left(\frac{C_3(n+1)}{n}\right)^{n+1}\left(r^{\frac{1}{n}+3} - a^{\frac{1}{n}+3}\right), \quad (17)$$

where

$$C_3 = \left(\frac{n}{1+n}\right)\left(\frac{|G| + i\omega\alpha\rho\exp(i\omega t)}{2m}\right)^{\frac{1}{n}}$$

and

$$C_4 = \alpha\exp(i\omega t).$$

Figure 3 shows the temperature profiles for three different times within the oscillation cycle. Clearly, the temperature profile changes within the oscillation cycle resulting from the changes in the velocity profiles. The temperature fields reach a maximum when $t = \pi$, (half a cycle). The shear thinning fluid shows a smaller temperature increase than that observed in the newtonian case. On the other hand, the temperature of the shear thickening fluid is higher that that calculated for the newtonian liquid. Since heating of the fluid results directly from viscous dissipation, it is to be expected that as the shear increases, liquid with constant or increasing viscosity with shear will attain higher temperatures. Figure 4 shows the effect of varying the power parameter n on the temperature profile for various cases of shear thinning and shear thickening fluids.

Now, to analyze the effect of the oscillation in the temperature increase of the fluid using the power-law model, the condition that $u_m \approx \alpha$ is relaxed. The temperature profile is then calculated for different values of α keeping the value of u_m constant. The effective temperature increase is quantified using the mean temperature $\overline{T}$, defined as

$$\overline{T} = \frac{\int_0^a T(r,t)\, dr}{\int_0^a dr}.$$

Hence, from Eq. (17), the mean temperature results in

$$\overline{T} - T_0 = C_1\left[-\frac{C_3 a^{\frac{1}{n}+3} + C_4 a^2}{6} + \frac{C_3 n^2}{3n+1}\left(\frac{a^{\frac{1}{n}+3}}{1+4n}\right)\right] + \frac{C_2 n^2}{1+3n}\left[\frac{C_3(n+1)}{n}\right]^{n+1}\left(\frac{a^{\frac{1}{n}+3}}{1+4n}\right). \quad (18)$$

Figure 5 shows the temperature increase as a function of the ration α/u_m for the three fluid cases. The temperature increases with the oscillation velocity. However, for high oscillation speeds a change of trend is observed. Trends similar to those observed here have also been reported by [?] for the case of non-oscilatory pipe flows.

4. Heat transfer coefficient

The dimensionless heat transfer coefficient is defined as

$$Nu = \frac{ha}{k} = \frac{a\left(\frac{\partial T}{\partial r}\right)_{r=a}}{\overline{T} - T_0}, \quad (19)$$

where h is the convective heat transfer coefficient and k is the thermal conductivity.

The mean temperature results has already been calculated and the temperature gradient at the wall can be obtained by

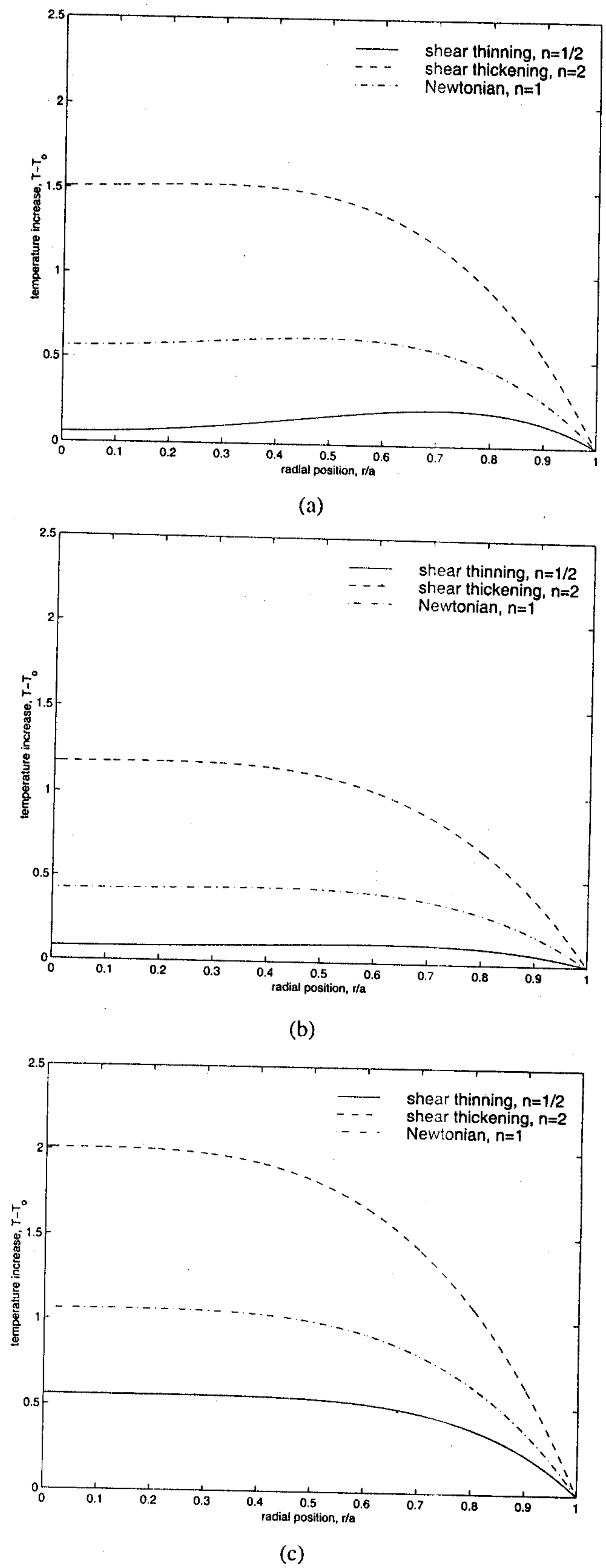

FIGURE 3. Temperature increase as a function of dimensionless radial position for power law fluids with: a) (———) shear thinning liquid, $n = 1/2$, b) (– – —) shear thickening liquid, $n = 2$, and c) (– · – · –·) newtonian liquid, $n = 1$. For $\omega\rho/\eta_0 = 1$, $\alpha/u_m = 1$, $\Delta T/\Delta z = 1$, and $m = 1$.

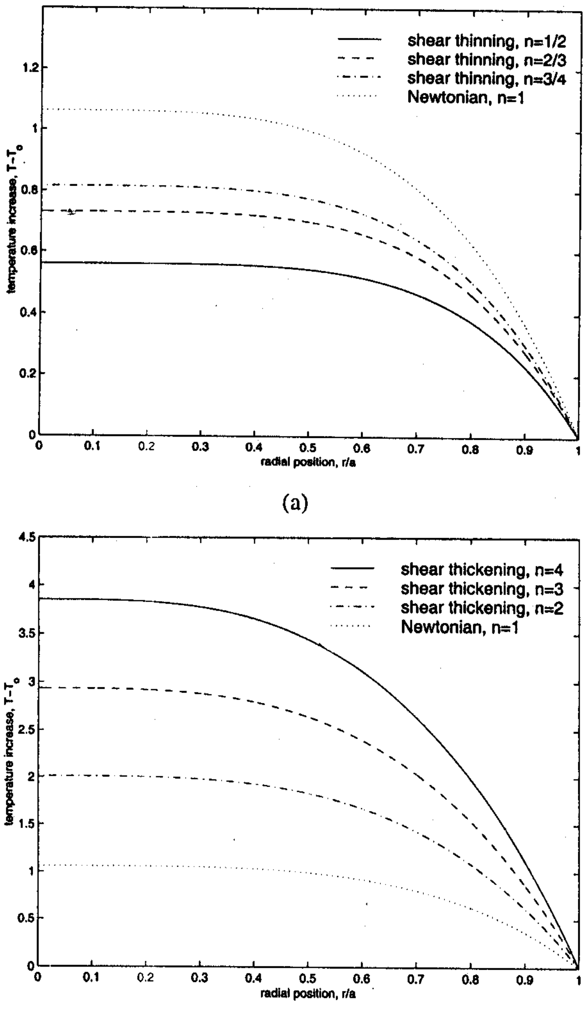

FIGURE 4. Temperature increase as a function of dimensionless radial position for shear thinning liquid and thickening liquids, for various values of the power parameter for $t = \pi$ and $\Delta T/\Delta z = 1$, and $m = 1$.

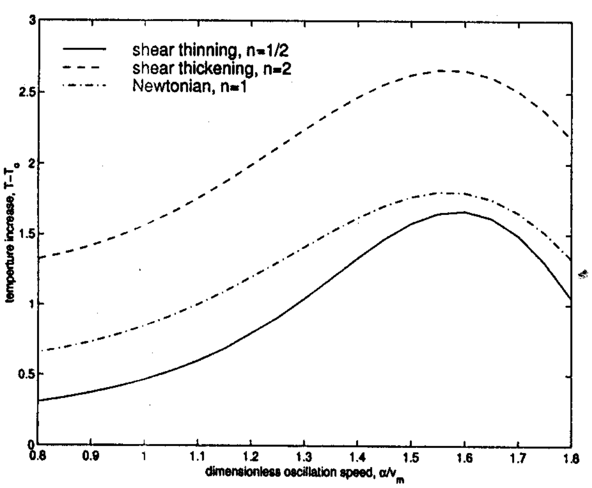

FIGURE 5. Temperature increase $\overline{T} - T_0$, as a function of the normalized oscillation velocity α/u_m, for $t = \pi$.

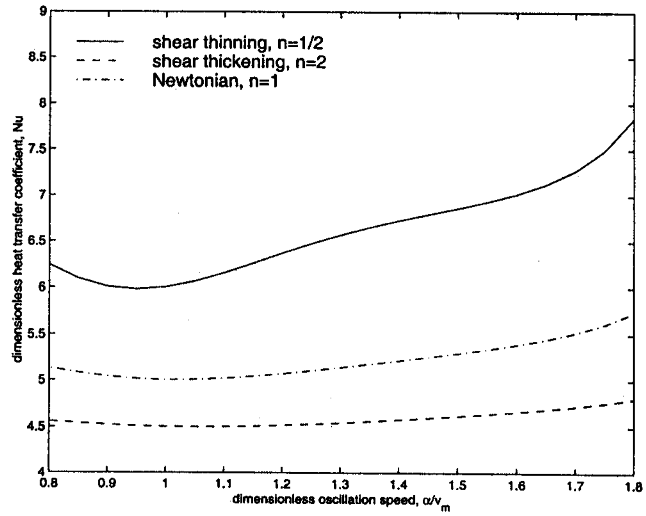

FIGURE 6. Heat transfer coefficient as a function of the oscillation velocity for laws liquid.

direct differentiation of the temperature profile [Eq. (17)] resulting in

$$\left(\frac{\partial T}{\partial r}\right)_{r=a} = C_1\left(\frac{C_3 a^{\frac{1}{n}+2} + C_4 a}{2} - \frac{C_3 n^2}{1+3n} a^{2+\frac{1}{n}}\right) - \frac{C_2 n}{1+3n}\left[\frac{C_3(n+1)}{n}\right]^{n+1} a^{2+\frac{1}{n}}. \quad (20)$$

Figure 6 shows the heat transfer coefficient as a function of the oscillating velocity for three cases studied. For the range showed, an increase of the heat transfer coefficient can be observed.

Aditionally, two limiting cases can be analyzed. For the case where the axial temperature gradient is zero ($C_1 = 0$), the Nusselt number, that corresponds to the case of pure viscous dissipation, reduces to

$$Nu_{\text{visc}} = \frac{1+4n}{n}.$$

Hence, it can be inferred that the heat transfer is only a function of the properties of the fluid and not of the oscillating characteristics of the flow. However, on the other hand, when only the convective contribution ($C_2 = 0$) is considered the heat transfer coefficient reduces to

$$Nu_{\text{conv}} = 3 + \frac{C_3 a^{\frac{1}{n}+3} + C_4 a^2}{1-(1+3n)(1+4n)\left(\frac{1}{6n^2} + \frac{C_4}{C_3 n^2 a^{\frac{1}{n}+1}}\right)},$$

which does not depend directly on the value of the axial temperature gradient $\Delta T/\Delta z$ but now depends on the oscillating characteristics of the flow (C_3 and C_4). Hence, to obtain an increase of the fluid temperature with increasing oscillation speed, the effect of the axial temperature gradient must be included.

5. Summary and conclusions

We have studied the temperature increase resulting from viscous dissipation in an oscillatory pipe flow. A variable viscosity inelastic "power law" model was used to characterize the rheology of the fluid. Both velocity and temperature fields were obtained for three fluid cases: newtonian, shear thinning and shear thickening fluids. Properties of unitary value were considered to obtain numerical results.

In general, it was found that the temperature increases with the power parameter and with the speed of oscillation. An effective heat transfer coefficient was also calculated which was observed to increase with the oscillation speed. To our knowledge, experimental measurements of the heat transfer in this flow configuration do not exist. We have recognized that there is a very small number of experimental studies that investigate the heat transfer processes in non-newtonian fluids.

Acknowledgments

The support of PAPIIT-DGAPA of UNAM through its grants IN503494 and IN103900, and CONACyT through its scholarship program and the grant 0073-A7026 are greatly acknowledged. JRH-V wishes to acknowledge the Instituto Tecnológico de Zacatepec for its support during the completion of his graduate studies.

---

* To whom correspondence should be addressed.

1. L. Greatz, *Ann. Phys. Chem.* **18** (1883) 79.
2. X. Yin and H.H. Bau, *J. Heat Trans.* **118** (1996) 482.
3. S.G. Etemad and H. Mujumdar, *Int. J. Heat Mass Trans.* **38** (1995) 2225.
4. O. Manero and B. Mena, *Rheol. Acta* **17** (1977) 573.
5. O. Manero, R. Valenzuela, and B. Mena, *Rheol. Acta* **17** (1978) 693.
6. B. Mena, O. Manero, and D.M. Binding, *J. Non-Newt. Fluid Mech.* **5** (1979) 427.
7. J.Casulli, J.R. Clermont, A. Von Ziegler, and B. Mena, *J. Polym. Eng. Sci.* **30** (1990) 1551.
8. J.R. Herrera-Velarde and B. Mena, *Rev. Mex. Fís.* **46** (2000) 1551.
9. W. Ostwald, *Kolloid-Z.* **36** (1925) 99.
10. R.B. Bird, R.C. Amstrong, and O. Hassager, *Dynamics of Polymeric Liquids*, Vol. 1: Fluid mechanics, (John Wiley & Sons, New York, 1977).