Is the Pauli Exclusive Principle an Independent Quantum Mechanical Postulate?

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Received 10 November 2001; accepted 10 November 2001

DOI 10.1002/qua.10276

ABSTRACT: Foundation of the Pauli exclusive principle is discussed. It is demonstrated that the indistinguishability principle is insensitive to the permutation symmetry of the wave function and cannot be used as a criterion for the verification of the Pauli exclusive principle. The heuristic arguments are given in favor that the existence in nature of only the nondegenerate permutation representations (symmetrical and antisymmetrical) is not occasional. As follows from our analysis of possible scenarios, the permission of degenerate permutation representations leads to contradictions with the concept of particle identity and their independence. Thus, the prohibition of degenerate permutation states by the Pauli exclusive principle follows from the general physical assumptions inside quantum theory, but the problem of spin–statistics connection is still open. It is pointed out that the Pauli exclusive principle and the Jahn–Teller effect have some similar features. Int J Quantum Chem 89: 268–276, 2002

Key words: Pauli principle; foundation of quantum mechanics; indistinguishability principle; spin–statistics connection; permutation symmetry

Introduction

Wolfgang Pauli established his principle before the creation of quantum mechanics (1925–1927). He arrived at formulation of this principle while trying to explain the regularities in the classification of atomic spectral terms in a strong magnetic field. In an article submitted for publication in January 1925, Pauli formulated his principle as follows [1]: "In an atom there cannot be two or more equivalent electrons for which the values of all four quantum numbers coincide. If an electron exists in an atom for which all of these numbers have definite values, then this state is '*occupied*.'" At that time, the fourth quantum number was not described by any model. Pauli called the property associated with it the "characteristic two-valuedness of the quantum properties of the electron which cannot be described classically" [2]. This nonclassic two-valued nature of an electron is now called *spin.* In anticipating the quantum nature of the magnetic moment of an electron before the creation of quantum mechanics, Pauli exhibited a striking intuition.

The first studies devoted to applying the newborn quantum mechanics to many-particle systems were those by Heisenberg [3] and Dirac [4]. In these studies, the Pauli principle, formulated as the prohibition for two electrons to occupy the same quantum state, was derived as a consequence of the antisymmetry of the wave function of the system of electrons. Dirac [4] came to the conclusion that the light quanta must be described by the symmetrical wave functions. He specially noted that a system of electrons cannot be described by the symmetrical wave function because the latter allow any number of electrons to occupy a quantum state.

Thus, with the creation of quantum mechanics, the prohibition on the occupation numbers of electron system states was supplemented by the prohibition of all types of permutation symmetry of electron wave functions except for antisymmetrical ones. Later, analysis of experimental data permitted to formulate the Pauli exclusion principle for all known elementary particles, namely,

> The only possible states of a system of identical particles possessing spin s are those for which the total wave function transforms upon interchange of any two particles as
>
> $$P_{ij}\Psi(1,\ldots,i,\ldots,j,\ldots,N) = (-1)^{2s}\Psi(1,\ldots,i,\ldots,j,\ldots,N), \quad (1)$$
>
> that is, it is symmetric for integer values of s and antisymmetric for half-integer s.

The exclusion principle also holds for the permutational symmetry of composite particle wave functions, for example, for nuclei. The latter consist from nucleons: protons and neutrons that are fermions because they have $s = 1/2$. Depending on the value of the total nuclear spin, one can speak of boson nuclei and fermion nuclei. The nuclei with an even number of nucleons have an integer value of the total spin and are characterized by the symmetrical total wave function. The nuclei with an odd number of nucleons have a half-integer value of the total spin and are characterized by the antisymmetrical wave function. The group-theoretical approach for finding quantum states of arbitrary systems allowed by the Pauli exclusive principle was elaborated in Refs. [5–7] (see also Ref. [8]).

All experimental data known to date agree with the Pauli exclusion principle. Several types of experiments on the search for possible small violations of the Pauli principle have been suggested in the literature [9–11]; a comprehensive discussion was presented at a recent conference [12] (see reports by Greenberg [13] and Gillaspy [14]). A test of the Pauli principle was made earlier by searching for γ quanta with energies around 20 MeV, which should be produced with transitions of nucleons in the ^{12}C nucleus from the 2p shell to the occupied 1s shell. This search has given a lower limit for the formation time of a "non-Pauli" nucleus $\tau \geq 2 \times 10^{20}$ years [15]. The search for X-ray radiation produced by the transition of an electron in the germanium detector to the occupied $(1s)^2$ shell, either in the case of the hypothetical spontaneous decay of an electron in this shell or in the case of violation of the Pauli principle, gave an even greater lower limit $\tau \geq 1.5 \times 10^{25}$ years [16].

The exclusion principle, first discovered by Pauli for electrons and later spread to all particles, was based on the analysis of experimental data. Pauli himself was never satisfied by this. In his Nobel Prize lecture [17], Pauli said: "Already in my initial paper, I especially emphasized the fact that I could not find a logical substantiation for the exclusion principle nor derive it from more general assumptions. I always had a feeling, which remains until this day, that this is the fault of some flaw in the theory." Let us stress that this was said in 1945, after the Pauli famous theorem [18] of the relation between spin and statistics. In this theorem, Pauli did not give a direct proof. He showed that due to some physical contradictions the second quantization operators for particles with integral spins cannot obey the fermion commutation relations, while for particles with half-integral spins they cannot obey the boson commutation relations. From this, Pauli concluded that particles with integral spin have to obey the Bose–Einstein statistics, while those with half-integral spin have to obey the Fermi–Dirac statistics.

Thus, according to the Pauli theorem, the connection between the value of spin and the permutational symmetry of a many-particle wave function, Eq. (1), follows if we assume that particles can obey only two types of commutation relations: boson or fermion relations. At that time, it was believed that it is really so. However, in 1953 Green [19] (see also Volkov [20]) showed that the more general, parabosonic and parafermionic trilinear commutation relations, satisfying all physical requirements and containing the boson and fermion commutation relations as particular cases, can be introduced. A corresponding parastatistics of rank p is characterized by the p-fold occupancy of a single-particle state. For $p = 1$, the parafermi statistics becomes identical to the Fermi–Dirac statistics, while the parabose statistics is reduced to the Bose–Einstein statistics [21].

As shown by Greenberg and Messiah [21], all known elementary particles are bosons or fermions. This conclusion does not spread on the quasiparticle case. In 1976, Kaplan [22] revealed that the parafermi statistics is realized for quasiparticles in a crystal lattice, but due to a periodical crystal field the trilinear commutation relations are modified by the quasiimpulse conservation law [22–24]; other applications of this modified parafermi statistics are shown in Refs. [25–27].

After 1940, numerous proofs of the spin–statistics theorem were published. All these proofs contain some explicit (or implicit) assumptions (see Duck and Sudarshan [28], reviewed by Wightman [29], and Proceedings [12]). As emphasized by Berry and Robbins [30], the relation between spin and statistics "cries out for understanding." We add that for composite particles the spin–statistics connection does not fulfill at all, for example, for the Cooper pairs [31].

In what follows, we will not discuss the relation between spin and statistics and focus on the symmetry restrictions of the Pauli exclusive principle. The point is that the Schrödinger equation is invariant under any permutation of identical particles. The Hamiltonian of an identical particle system commutes with permutation operators,

$$[P, H]\_ = 0. \tag{2}$$

As a result, the solutions of the Schrödinger equation may belong to any representation of the permutation group, including the degenerate ones. However, according to the exclusion principle, a system of identical particles can be only in those states that are not degenerate with respect to permutations; that is, in a symmetrical or antisymmetrical state; all other types of symmetry are forbidden. The question might be asked whether this limitation on the solutions of the Schrödinger equation follows from the fundamental principles of quantum mechanics or it is an independent principle?

In the next sections, we discuss the possible answers to this question, developing some ideas from our previous publications [32, 33].

Indistinguishability of Identical Particles and the Symmetry Postulate

There are two points of view on the problem of independency of the Pauli exclusive principle from other fundamental quantum mechanical postulates. Some physicists, including one of the founders of quantum mechanics (Dirac [34]; see also Shiff [35] and Messiah [36]), believe that there are no laws in nature that forbid the existence of particles described by wave functions with more complicated permutation symmetry than those of bosons and fermions, and that the existing limitations are only due to the specific properties of the known elementary particles. Messiah [36] even introduced the term *symmetry postulate* to emphasize the primary nature of the constraint on the allowed types of the wave function permutation symmetry. By using the Schur lemma, Messiah and Greenberg [37] have shown that the existence of permutation degeneracy should not introduce additional uncertainty into the characteristics of a state. This also follows directly from the Wigner–Eckart theorem generalized for the permutation group (see Eq. (4.60) in Ref. [8]), namely, the matrix element of an operator L, which is symmetrical in all the particles, can be presented as

hat {L
hat {L

$$\langle \Psi_r^{[\lambda]} | \hat{L} | \Psi_{\mathrm{bar}\,\{r}^{[\lambda]} \rangle = \delta_{r\bar{r}} \langle \Gamma^{[\lambda]} \| \hat{L} \| \Gamma^{[\lambda]} \rangle \hat{L}, \tag{3}$$

where index r labels the basic functions of the representation $\Gamma^{[\lambda]}$ of the permutation group. The double vertical line in the right side of this formula means that the matrix element is independent of the basic function index. Thus, the expectation value of operator L is the same for all functions belonging to the degenerate state.

Another point of view is that the symmetry postulate is not an independent principle but can be derived from the fundamental principles of quantum mechanics, in particular from the principle of indistinguishability of identical particles. The typical argumentation presented in some textbooks and monographs [38–40] follows.

From the requirement that the states of a system obtained by permutations of identical particles must all be physically equivalent, one concludes that the change in the wave function resulting from the transposition of any two identical particles should only cause multiplication by an insignificant phase factor:

$$P_{12}\Psi(x_1, x_2) = \Psi(x_2, x_1) = e^{i\alpha}\Psi(x_1, x_2), \quad (4)$$

where α is a real constant and x is the set of spatial and spin variables. One more application of the permutation operator gives

$$\Psi(x_1, x_2) = e^{i2\alpha}\Psi(x_1, x_2) \quad (5)$$

or

$$e^{2i\alpha} = 1 \quad \text{and} \quad e^{i\alpha} = \pm 1. \quad (6)$$

Because all particles are assumed to be identical, the wave function should change in exactly the same way under transposition of any pair of particles, that is, it should be either totally symmetrical or totally antisymmetrical.

The incorrectness of this proof is in the following: Equation (4) is valid only for 1-D representations. The common belief that wave functions describing the same physical state may differ by no more than a phase factor is evidently not true. For instance, according to Eq. (3), the values of the physical quantities characterizing a system of identical particles are the same for all functions belonging to the same irreducible representation, and all these different analytic functions describe the same physical state. According to the group theory, the application of a group operation to one of basic functions, belonging to some degenerate representation, transforms it as a linear combination of basic functions. By requiring that under permutations, the wave function must change by no more than a phase factor, one actually *postulates* that the representation of the permutation group is one-dimensional (1-D). Thus, the proof [38–40] is based on the initial statement, which then is proved as a final result.

In the above proof, there is an additional incorrectness: The indistinguishability principle is directly related to the behavior of the wave function. However, because the wave function is not an observable, the indistinguishability principle is related to it only indirectly via the expressions for measurable quantities. A rigorous proof should be based on a rigorous formulation of the indistinguishability principle for identical particles. One possible formulation is the following: All observable quantities are invariant under the permutations of identical particles and, vice versa, the permutations of identical particles cannot be observed.

Because in quantum mechanics the physical quantities are expressed as bilinear forms of wave functions, the indistinguishability principle requires the invariance of these bilinear forms and can be formulated as [41]

hat {L
hat {L

$$P\langle\Psi|\hat{L}|\Psi\rangle = \langle\Psi|\hat{L}|\Psi\rangle. \quad (7)$$

Often, one limits oneself to the requirement that the probability of a given configuration of a system of identical particles must be invariant under permutations [42, 43]:

$$P|\Psi(x_1, \ldots, x_N)|^2 = |\Psi(x_1, \ldots, x_N)|^2. \quad (8)$$

For a function to satisfy Eq. (8), it is sufficient that under permutations it would change as

$$P\Psi(x_1, \ldots, x_N) = e^{i\alpha_p(x_1, \ldots, x_N)}\Psi(x_1, \ldots, x_N), \quad (9)$$

that is, unlike the case of the requirement of condition (4), in the general case, the phase is a function of coordinates and the permutation; and Eq. (5) evidently does not hold.

In a degenerate state, the system can be described with the equal probability by any one of the basic vectors of the degenerate state. As a result, we can no longer select a pure state (the one that is described by the wave function) and should regard the degenerate state as a mixed one, where each basis vector enters with the same probability. We must sum both sides of Eqs. (7) and (8) over all wave functions that belong to the degenerate state. As showed by von Neumann [44], the diagonal element of the density matrix for a degenerate state has the form

$$D(x_1, \ldots, x_N; x_1, \ldots, x_N) = \frac{1}{f_\lambda}\sum_{r=1}^{f_\lambda} \Psi_r^{[\lambda]}(x_1, \ldots, x_N)^*\Psi_r^{[\lambda]}(x_1, \ldots, x_N), \quad (10)$$

where expression (10) is written for the case of f_λ-dimensional representation $\Gamma^{[\lambda]}$ of the permutation group π_N. The possibility of expressing the density matrix through only one of the functions implies that the degeneracy with respect to permutations can be eliminated. However, the latter cannot be achieved without violating the identity of the particles.

It is not difficult to check that for every representation $\Gamma^{[\lambda]}$ of the permutation group π_N the probability density, Eq. (10), is a group invariant:

$$PDP^{-1} = D \qquad \text{for all} \quad P \in \pi_N. \tag{11}$$

From this follows that the probability density obeys the indistinguishability principle even in the case of multidimensional representations of the permutation group. Thus, the indistinguishability principle is insensitive to the symmetry of the wave function and cannot be used as a criterion for selecting the correct symmetry.

It is worth noting that from the discussion above it does not follow that the symmetry of wave function is not significant and one can perform quantum mechanical study using only the density matrix, which, as we have shown, does not depend upon the symmetry of the wave function. In reality, this symmetry controls the atomic and molecular states allowed by the Pauli principle. For instance, in atomic spectroscopy it is known that in the $(np)^2$ electronic shell only ^{1}S, ^{3}P, and ^{1}D states are realized. This follows directly from the antisymmetry of the total electronic function. The symmetry of a many-particle state and its multiplicity are completely dictated by the symmetry of the wave function attributed to this state [8]. So, the widespread application of the density functional theory based on the Kohn–Sham equation does not mean that the concept of wave function in quantum mechanics lost its importance.

Although the Pauli exclusive principle cannot be rigorously derived from other quantum mechanical postulates, there are some heuristic arguments indicating that the description of an identical particle system by degenerate representations of the permutation group leads to some contradictions with the concept of the particle identity and their independency. In the next section, we discuss these arguments in detail.

Contradictions with the Concept of Particle Identity in the Permutation Degenerate States

In this section, we discuss the properties of a quantum mechanical system of identical particles that does not obey the symmetrization postulate and can be in states with all possible permutation symmetries. To the best of our knowledge, Steinmann [45] was the first who considered the identical particle system in a degenerate permutation state. He considered the so-called triangular representation of the permutation group π_3 characterized by the Young diagram $[\lambda] = [21]$[1] and came to the conclusion that particles in the representation $\Gamma^{[21]}$ are distinguishable because if we performed the reduction $\pi_3 \to \pi_2$, the two-particle state will be a mixture of symmetrical and antisymmetrical states. The Steinmann arguments were criticized by Hartle and Taylor [46], who showed that, in general, the indistinguishability is preserved in the case of degenerate representations $\Gamma^{[\lambda]}$. They concluded that there are no theoretical reasons against an existence of degenerate permutation states in quantum mechanics although it is disagreeable. The same conclusion was made earlier by Casher et al. [47], who labeled this situation unpalatable.

As shown in the previous section, the indistinguishability principle is insensitive to the permutation symmetry and is satisfied by functions belonging to the degenerate permutation states. In this sense, the critique of Steinmann arguments [45] by Hartle and Taylor [46] was correct. Nevertheless, the general conclusion [46, 47] that there are no theoretical prohibitions on the existence of some unknown identical particle systems in degenerate permutation states is not obvious and it is worth revising this conclusion. As we will see below, the assumption in Refs. [46, 47] that the triangular representation for three particles can originate from the state of two particles described by the linear combination of symmetrical and antisymmetrical functions leads to distinguishable particles [see Eq. (27)].

Let us consider a quantum mechanical system of identical elementary particles without the restrictions imposed by the symmetrization postulate and base our study on the Hartree–Fock one-particle

[1] The classification of irreducible representations of permutation groups according to the Young diagrams can be found in Ref. [8].

approximation. In this case, the states of a system of identical particles with the number of particles not conserved can be presented as vectors in the Fock space F [48]. The latter is a direct sum of spaces $F^{(N)}$ corresponding to a fixed number of particles N:

$$F = \sum_{N=0}^{\infty} F^{(N)}. \tag{12}$$

Each of the space $F^{(N)}$ can be presented as a direct product of one-particle spaces f:

$$F^{(N)} = \underbrace{f \otimes f \otimes K \otimes f}_{N}. \tag{13}$$

The basic vectors of $F^{(N)}$ are the product of one-particle vectors $|\nu_k(k)\rangle$ belonging to space f; k in the parentheses denotes the set of particle spin and space coordinates,

$$|\xi^{(N)}\rangle = |\nu_1(1)\rangle|\nu_2(2)\rangle \ldots |\nu_N(N)\rangle. \tag{14}$$

For simplicity, let us consider the case where all vectors in Eq. (14) are different. There will be no qualitative changes in the results if some of the vectors $|\nu_k\rangle$ coincide. One can produce N! new vectors by applying to the vector (14) N! permutations of the particle coordinates. These new vectors also belong to $F^{(N)}$ and form in it a certain invariant subspace that is reducible. The N! basic vectors of the latter, $P|\xi^{(N)}\rangle$, make up the regular representation of the permutation group π_N. As is known, the regular representation is decomposed into irreducible representations, each of which appears a number of times equal to its dimension. The space $\varepsilon^{(N)}$ falls into the direct sum

$$\varepsilon_{\xi}^{(N)} \overset{\dot{\iota}}{=} \sum_{\lambda_N} f_{\lambda_N} \varepsilon_{\xi}^{[\lambda_N]}, \tag{15}$$

where $\varepsilon_{\xi}^{[\lambda_N]}$ is an irreducible subspace of dimension f_λ drawn over the basic vectors $|[\lambda_N]r\rangle$ ($[\lambda_N]$ is a Young diagram with N boxes). These vectors are constructed of nonsymmetrized basic vector $|\xi^{(N)}\rangle$ by using the Young operators $\omega_{rt}^{[\lambda_N]}$ [8],

$$|[\lambda_N]rt\rangle = \omega_{rt}^{[\lambda_N]}|\xi^{(N)}\rangle = \left(\frac{f_\lambda}{N!}\right)^{1/2} \sum_P \Gamma_{rt}^{[\lambda_N]}(P)\, P|\xi^{(N)}\rangle, \tag{16}$$

where $\Gamma_{rt}^{[\lambda_N]}(P)$ are the matrix elements of representation $\Gamma^{[\lambda_N]}$ and index t distinguishes between the bases in accordance with the decomposition of $\varepsilon_{\xi}^{(N)}$ into f_λ invariant subspaces and describes the symmetry under permutations of the particle vector.

Thus, a space with a fixed number of particles can always be divided into irreducible subspaces $\varepsilon_{\xi}^{[\lambda_N]}$, each of which is characterized by a certain permutation symmetry given by a Young diagram with N boxes. The symmetry postulate requires that the basis vectors of a system of N identical particles belong to one of the subspaces characterized by irreducible 1-D representations, either $[N]$ or $[1^N]$. All other subspaces are "empty." Let us examine the situation that arises when no symmetry constraints are imposed.

As is known, one of the consequences of the different symmetry of state vectors for bosons and fermions is the dependence of the energy of system on the particle statistics. For the same law of dynamic interaction, the so-called exchange terms enter the expression for the energy of fermion and boson system with opposite signs. Let us obtain the expression for the energy of a system of particles belonging to an irreducible subspace $\varepsilon_{\xi}^{[\lambda_N]}$ with an arbitrary Young diagram $[\lambda_N]$.

The energy of the system in a degenerate state is

$$E = \mathrm{Tr}(HD), \tag{17}$$

where D is the density operator defined, similarly to Eq. (10), as

$$D_t = \frac{1}{f_\lambda} \sum_{r=1}^{f_\lambda} |[\lambda]rt\rangle\langle[\lambda]rt|. \tag{18}$$

We assume that the Hamiltonian includes only one- and two-particle interaction operators:

$$H = \sum_i h_i + \sum_{i<j} g_{ij}. \tag{19}$$

The calculation of the trace over the functions with symmetry $[\lambda_N]$ yields

$$E_t^{[\lambda]} = \frac{1}{f_\lambda} \sum_{r=1}^{f_\lambda} \langle[\lambda]rt|H|[\lambda]rt\rangle. \tag{20}$$

The matrix element in Eq. (20) has been calculated in Ref. [49] in a general case of nonorthogonal one-particle vectors. In the case where all vectors in Eq. (14) are different and orthogonal, one gets

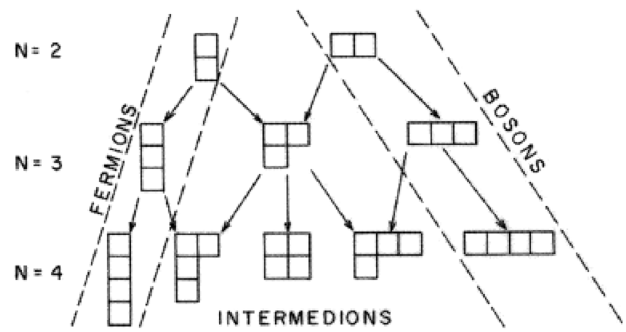
$$E_t^{[\lambda]} = \sum_a \langle \nu_a | h | \nu_a \rangle + \sum_{a<b} [\langle \nu_a \nu_b | g | \nu_a \nu_b \rangle + \Gamma_{tt}^{[\lambda]}(P_{ab}) \langle \nu_a \nu_b | g | \nu_b \nu_a \rangle], \quad (21)$$

where $\Gamma_{tt}^{[\lambda]}(P_{ab})$ is the diagonal matrix element of the transposition of vectors $|\nu_a\rangle$ and $|\nu_b\rangle$ in the product (14).

Thus, the energy of system in a state with symmetry corresponding to a Young diagram [λ] depends not only on [λ] but also on a type of the Young tableaus t characterizing the symmetry of the basic functions according to the permutations of the one-particle vectors, index t. Because the transitions between f_λ irreducible subspaces, characterized by index t, are allowed, there are f_λ split levels with symmetry [λ].

Only exchange terms in Eq. (21) depend upon the symmetry of states. For 1-D representations, $\Gamma_{tt}^{[\lambda]}(P_{ab})$ do not depend on the number of particles: $\Gamma^{[N]}(P_{ab}) = 1$ and $\Gamma^{[1^N]}(P_{ab}) = -1$ for all P_{ab} and N. For multidimensional representations, the matrix elements $\Gamma_{tt}^{[\lambda_N]}(P_{ab})$ depend on $[\lambda_N]$ and P_{ab}; in general, they are different for different pairs of identical particles.[2]

It is natural that a different symmetry of state vector leads to a different expression for the energy, as follows from Eq. (21). The transitions between states with different symmetry $[\lambda_N]$ are strictly forbidden. Each state of an N-particle system with different $[\lambda_N]$ has a different analytic formula for its energy. So, we may conclude that each type of symmetry $[\lambda_N]$ corresponds to a certain kind of particle with statistics determined by this permutation symmetry. On the other hand, the classification of state with respect to the Young diagrams $[\lambda_N]$ is connected exclusively with the identity of particles. Therefore, it must be some additional inherent particle characteristic that establishes for the N-particle system to be in a state with a definite permutation symmetry, like integer and half-integer values of particle spin for bosons and fermions. Let us call these hypothetical particles characterized by the degenerate representations of permutation group *intermedions,* implying that they obey some intermediate between fermion and boson statistics (it has not necessary to be the parastatistics).



For bosons and fermions, there are two nonintersecting chains of irreducible representations: $[N] \to [N+1]$ and $[1^N] \to [1^{N+1}]$, respectively, and the energy expression for each type of particle has the same analytic form that does not depend on the number of particles in a system. The situation changes drastically if we put into consideration the degenerate representations. The number of different statistics depends on the number of particles in a system and rapidly increases with N. As we show below, for degenerate representations we cannot select any nonintersecting chains, as in the fermion and boson cases.

Let us trace down the genealogy of irreducible subspaces $\varepsilon_\xi^{[\lambda_N]}$ for $N = 2$–4. According to the above genealogy, the intermedion particles with a definite $[\lambda_N]$ in the Nth generation can originate from particles of different kinds $[\lambda_{N-1}]$ in the $(N-1)$th generation, even from fermions or bosons (in the special case $[\lambda_3] = [21]$, it originates from both $[1^2]$ and [2]). The physical picture in which adding one particle changes properties of all particles cannot correspond to a system of *independent* identical particles (although it cannot be excluded for some quasiparticle systems where we have not an independency of quasiparticles; see Refs. [22, 23]).

If we consider an ideal gas, it is evident that adding a particle identical to a system of N identical particles cannot change the properties of new $(N+1)$-particle system. But, the interaction of identical particles does not change the permutation symmetry of a noninteracting particle system [41], as the interaction operator is invariant according to permutations of identical particles. Thus, the scenario in which each symmetry type $[\lambda_N]$ corresponds to a definite particles statistics contradicts the concept of particle identity and their independency from each other.

[2] The matrices of transpositions for all irreducible representations of groups π_2–π_6 are presented in Ref. [8], Appendix 5.

Let us consider another virtual possibility and begin from the ordinary fermions. As is well known in quantum mechanics of identical particles, in the absence of spin interactions the total spin S is a good quantum number and labels the energy levels of the system. The symmetry of the coordinate wave function depends on the value of S, which causes the dependence on S of the total energy of the system regardless of the dynamic interaction law. The symmetry of the coordinate wave function corresponds to some coordinate Young diagram $[\lambda_N]_{coord}$ (uniquely connected with the spin Young diagram $[\lambda_N]_{spin}$) that can belong to a degenerate representation of the group π_N (see [8]). Therefore, the system of fermions may be described by the degenerate permutation representations; however, the latter correspond not to the total wave function but to its factorized parts. The total wave function is completely antisymmetrical, in accordance with the Pauli exclusive principle.

Hence, we should consider the possibility of realization of degenerate representations $[\lambda_N]$ for the *total* wave function including *all* degrees of freedom of the particles under consideration. As we showed in the scenario considered first, the assumption that for the total wave function all possible $[\lambda_N]$ can be allowed leads to contradictions with the concept of particle identity and their independency. Therefore, we have to consider the possibility that for some type of intermedions for each fixed N only one type of $[\lambda_N]$ is permitted (as is in the case of fermions, $[1^N]$, and bosons, $[N]$). But, in this scenario we meet with a serious problem more evident for a two-particle case.

For $N = 2$, regardless of the physical nature of particles, only symmetrical (boson) and antisymmetrical (fermion) representations exist.* We have no other option but to assume that the symmetry of a two-intermedion system coincides with the two-fermion or two-boson system. The point is that, contrary to the statement in Refs. [46, 47], the two-particle wave function cannot be described by some coherent superposition

$$\Psi_n(x_1, x_2) = c_1\Psi^{[2]}(x_1, x_2) + c_2\Psi^{[1^2]}(x_1, x_2) \quad (26)$$

because this superposition describes distinguishable particles. In fact,

$$P_{12}\Psi_n(x_1, x_2) = c_1\Psi^{\lfloor 2 \rfloor}(x_1, x_2) - c_2\Psi^{\lfloor 1^2 \rfloor}(x_1, x_2) \neq \Psi_n(x_1, x_2). \quad (27)$$

The situations labeled in Refs. [46, 47] as "disagreeable" or "unpalatable" in reality are physically forbidden. So, we have to consider two particles not in a mixed but in the pure fermion or boson state. In this case, the addition of the third particle, identical to the two others, changes the fermion (or boson) statistics on the intermedion statistics with $[\lambda_3] = [21]$, and this take place even in an ideal gas of intermedions. Again, we came to the contradiction with the concept of particle identity and their independence.

All contradictions discussed above are resolved if only the 1-D irreducible representations of the permutation group are permitted, as follows from the Pauli exclusive principle. Thus, the existence in nature of only symmetrical and antisymmetrical types of permutation symmetry is not occasional. It is intimately connected with the identity of particles.

Conclusions

Despite more than 75 years of studies of the Pauli exclusive principle and spin–statistics connection, we still do not have a rigorous theoretical ground for it. As demonstrated, the indistinguishability principle is insensitive to the permutation symmetry of wave function and cannot be used for the verification of the Pauli exclusive principle. Experimental data and checking known to date confirm the Pauli exclusive principle; all elementary particles belong only to one of two statistics: fermion or boson statistics (for quasiparticles, this is not true; see Refs. [22, 23]).

Although the Pauli exclusive principle has no rigorous theoretical proof and follows from experiment, the existence in nature of only the nondegenerate permutation states is not occasional. For the different virtual scenarios discussed, the permission for an identical particle system to be in a degenerate permutation state leads to contradictions with the concept of particle identity and their independence.

Thus, if we reduce the Pauli exclusive principle only to the prohibition of all types of the permutation symmetry, except completely symmetrical and antisymmetrical ones, it can be considered as a consequence of general physical assumptions in-

*__Note added in proof.__ This is not true for quasiparticles, e.g. for anyons or any objects with so-called fractional statistics. The latter are not independent objects and the permutation group cannot be applied to the Hilbert space of anyons, see Chen Y.-H. et al., Int J Mod Phys 1989, B3, 1001.

side quantum theory. But, the problem of spin–statistics connection is still open.

There is an interesting similarity between the Pauli exclusive principle and the Jahn–Teller effect [50]. According to the latter (see [51, 52]), molecules (nonlinear) and crystals are not stable if the ground electronic state is degenerate. Due to the vibronic interactions, the symmetry of the ground-state conformation lowers down to a symmetry for which the system has the nondegenerate ground state. It belongs to a 1-D representation of a point (space) group of symmetry. The same follows from the Pauli exclusive principle for the permutation group: Only the 1-D (nondegenerate) representations of the permutation group are realized. This similarity in two different, at first glance, physical phenomena is worthwhile for special study.

ACKNOWLEDGMENTS

The author is grateful to Isaak Bersuker and Othmar Steinmann for useful and stimulating discussions. This study was partly supported by CONACyT (Mexico) Grant .32227-E.

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