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Harmonically trapped ideal quantum gases

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Abstract. We solve the problem of a Bose or Fermi gas in d-dimensions trapped by $\delta \leq d$ mutually perpendicular harmonic oscillator potentials. From the grand potential we derive their thermodynamic functions (internal energy, specific heat, etc.) as well as a generalized density of states. The Bose gas exhibits Bose-Einstein condensation at a nonzero critical temperature T_c if and only if $d+\delta > 2$, along with a jump in the specific heat at T_c if and only if $d+\delta > 4$. Specific heats for both gas types precisely coincide as functions of temperature when $d+\delta = 2$. The trapped system behaves like an ideal free quantum gas in $d+\delta$ dimensions. For $\delta = 0$ we recover all known thermodynamic properties of ideal quantum gases in d dimensions, while in 3D for $\delta = 1$, 2 and 3 one simulates behavior reminiscent of quantum *wells, wires* and *dots*, respectively. Good agreement is found between experimental critical temperatures for the trapped boson gases $^{87}_{37}$Rb, $^{7}_{3}$Li, $^{85}_{37}$Rb, $^{4}_{2}$He, $^{41}_{19}$K and the known theoretical expression which is a special case for $d = \delta = 3$, but only moderate agreement for $^{27}_{11}$Na and $^{1}_{1}$H.

PACS. 05.30.Fk Fermion systems and electron gas – 05.30.Jp Boson systems – 03.75.Fi Phase coherent atomic ensembles; quantum condensation phenomena – 05.70.Ce Thermodynamic functions and equations of state

1 Introduction

Ultra-cooled bosonic clouds trapped in a harmonic oscillator (HO) external potential mimic the behavior of bosons confined by realistic potentials as in opto-magnetic traps in the region of small oscillations. Bose-Einstein condensation (BEC) has been now observed with $^{87}_{37}$Rb [1], $^{27}_{11}$Na [2], $^{7}_{3}$Li [3], $^{1}_{1}$H [4], $^{85}_{37}$Rb [5], $^{4}_{2}$He [6] and $^{41}_{19}$K [7] neutral bosonic atoms, the upper and lower prefixes being the nuclear mass (number of nucleons in the nucleus) and proton numbers, respectively.

BEC has also been observed in lower dimensions. Görlitz *et al.* [8] report BEC of $^{23}_{11}$Na atoms in 1D or 2D; Schreck *et al.* [9] observe it with $^{7}_{3}$Li atoms in 1D; and Burger *et al.* [10] study the phase transition in a cloud of $^{87}_{37}$Rb atoms in quasi-2D.

Trapped quantum gases have been discussed in general by several authors [11–20]. The first calculation of the properties of a Bose gas in an isotropic harmonic trap was reported by de Groot *et al.* [11]; Bagnato *et al.* [12] reported theoretical thermodynamic properties of a Bose gas confined by a generic power-law potential trap; Ketterle *et al.* [13] and Pathria [16] considered the BEC of a finite number ($\sim 10^5$) of particles confined in a 3D HO and concluded that the thermodynamic-limit approximation is good; Petrov *et al.* [17,18] discuss BEC in quasi-2D trapped gases and study phase-coherence properties in 3D. For a review of BEC in trapped dilute Bose gases, see reference [20].

Trapped Fermi gases have also gained interest as possible precursors of a paired-fermion condensate at lower temperatures [21–23], and have been studied experimentally in ultracold fermionic clouds, *e.g.*, with $^{40}_{19}$K neutral atoms in opto-magnetic traps [24–27].

Finally, the discovery of the quasi-2D superconductors such as the cuprates [28–30] or the quasi-1D superconductors such as the organo-metallics (or Bechgaard salts) [31–33] have also motivated studying confinement of quantum gases.

In this paper we describe boson or fermion HO trapping in order to better understand these lower-dimensional structures. Since the system dimensionality modifies the nature of BEC, or even precludes it, we seek an exact and complete solution in the thermodynamic limit to the non-interacting Bose or Fermi gas problem in d-dimensions constrained by a number δ of mutually-perpendicular HO external potentials. We show that it is possible to map this problem into that of a free gas but in a higher $d+\delta$ dimensionality. For example, confinement [34–37] in 3D by a 1D

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HO potential collapses the system to a quasi-2D "slab" reminiscent of a quantum *well.* Confinement by a 2D (or 3D) HO potential leads to a quantum *wire*-like quasi-1D (or quantum *dot*-like quasi-0D) system.

In Section 2 we calculate the thermodynamic (or grand) potential for the non-interacting Bose or Fermi gas in d-dimensions with δ mutually-perpendicular HO traps. In Section 3 we deduce the thermodynamic properties of these systems and extract a generalized density of states, find their thermodynamic limit, and exhibit their mapping to free gases in higher dimensions. In Section 4 we specialize to a trapped boson gas and obtain its critical BEC temperature, its condensate fraction and specific heat cusp or jump singularity. We also summarize findings for a 3D boson gas trapped by 1, 2 or 3 HO's. In Section 5 we specialize to a trapped fermion gas. Section 6 contains our conclusions.

2 A d-dimensional quantum gas trapped by $\delta \leq$ d HOs

We consider a d-dimensional noninteracting boson or fermion gas trapped by $\delta = 1, 2, ..., d$ mutually-perpendicular harmonic-oscillator potentials, the particles otherwise moving freely in the remaining $d - \delta$ directions. The Hamiltonian for a single boson or fermion of mass m is $H = \sum_{i=1}^{d} p_i^2/2m + \frac{1}{2}\, m\, \omega^2 \sum_{j=d-\delta+1}^{d} r_j^2$ and its eigenvalues are

$$\varepsilon_{\{n_i,\nu_j\}} = \frac{2\pi^2\hbar^2}{mL^2}\sum_{i=1}^{d-\delta} n_i^2 + \hbar\omega\sum_{j=1}^{\delta}(\nu_j + 1/2) \qquad (1)$$

where L is the size of the "box" associated with the $d - \delta$ free dimensions and where $n_i = 0,\ \pm 1,\ \pm 2,\ ...$ while $\nu_j = 0,\ 1,\ 2,\ ...$ Since $k_i \equiv (2\pi/L)n_i$ and defining the variable $l_j \equiv \hbar\omega\nu_j$, (1) can then be rewritten as

$$\varepsilon_{\{k_i,l_j\}} = \frac{\hbar^2}{2m}\sum_{i=1}^{d-\delta} k_i^2 + \sum_{j=1}^{\delta} l_j + \hbar\omega\delta/2. \qquad (2)$$

The grand potential $\Omega(T,V,\mu)$ can then be written in generalized form (see p. 134 of [38]) as

$$\begin{aligned}\Omega(T,V,\mu) &= U - TS - \mu N \\ &= \delta_{a,-1}\Omega_0 - \frac{k_{\rm B}T}{a}\sideset{}{'}\sum_{\{k_i,l_j\}} \ln\left[1 + a\mathrm{e}^{-\beta(\varepsilon_{\{k_i,l_j\}}-\mu)}\right], \qquad (3)\end{aligned}$$

where the primed summation sign excludes the $k_i = 0 = l_j$ terms in the boson case. Here $V \equiv L^{d-\delta}x_0^{2\delta}$ is a confinement volume with $x_0 \equiv \sqrt{\hbar/m\omega}$ the oscillator length parameter, U the internal energy, T the absolute temperature, S the entropy, μ the chemical potential, N the number of particles, $a = -1$ for bosons, $a = 1$ for fermions and $a \to 0$ in the classical case, δ is the Kronecker delta function, and $\beta \equiv 1/k_{\rm B}T$. In the case of a Bose gas it is convenient to separate out in the sum (3), the lowest energy state from the excited states. We thus defined $\Omega_0 \equiv -(k_{\rm B}T/a)\ln[1+a\mathrm{e}^{-\beta(\hbar\omega\delta/2-\mu)}]$ corresponding to the ground state contribution to the grand potential. Using the logarithm expansion $\ln(1+x) = -\sum_{l=1}^{\infty}(-x)^l/l$ valid for $|x| < 1$, (3) becomes

$$\begin{aligned}\Omega(T,V,\mu) &= \delta_{a,-1}\Omega_0 + \frac{k_{\rm B}T}{a}\sideset{}{'}\sum_{\{k_i,l_j\}}\sum_{l=1}^{\infty}\frac{(-a\mathrm{e}^{-\beta(\varepsilon_{\{k_i,l_j\}}-\mu)})^l}{l} \\ &= \delta_{a,-1}\Omega_0 + \frac{k_{\rm B}T}{a}\sum_{l=1}^{\infty}\frac{(-a\mathrm{e}^{\beta\mu})^l}{l} \\ &\quad\times \sideset{}{'}\sum_{\{k_i,l_j\}} \mathrm{e}^{-\beta l\left[(\hbar^2/2m)\sum_{i=1}^{d-\delta}k_i^2+\sum_{j=1}^{\delta}l_j+\hbar\omega\delta/2\right]}.\end{aligned} \qquad (4)$$

In the continuous limit where $\hbar^2/mL^2 \ll k_{\rm B}T$ and $\hbar\omega \ll k_{\rm B}T$ (level spacing negligible compared to temperature), the summations over k_i and l_j can be approximated by integrals, namely $\sum_{k_i} \to (2\mathsf{s}+1)(L/2\pi)^{d-\delta}\int \mathrm{d}^{d-\delta}k_i$ and $\sum_{l_j} \to (2\mathsf{s}+1)(\hbar\omega)^{-\delta}\int \mathrm{d}^{\delta}l_j$. Thus

$$\begin{aligned}\Omega(T,V,\mu) &= \delta_{a,-1}\Omega_0 + \frac{k_{\rm B}T(2\mathsf{s}+1)(L/2\pi)^{d-\delta}(\hbar\omega)^{-\delta}}{a} \\ &\times \sum_{l=1}^{\infty}\frac{(-a\mathrm{e}^{\beta\mu})^l}{l}\int_{-\infty}^{\infty}\mathrm{d}k_1\ \mathrm{e}^{-\beta l(\hbar^2/2m)k_1^2} \\ &\times \int_{-\infty}^{\infty}\mathrm{d}k_2\ \mathrm{e}^{-\beta l(\hbar^2/2m)k_2^2}...\int_{-\infty}^{\infty}\mathrm{d}k_{d-\delta}\ \mathrm{e}^{-\beta l(\hbar^2/2m)k_{d-\delta}^2} \\ &\times \int_0^{\infty}\mathrm{d}l_1\ \mathrm{e}^{-\beta l(l_1+1/2)}...\int_0^{\infty}\mathrm{d}l_\delta\ \mathrm{e}^{-\beta l(l_\delta+1/2)}.\end{aligned} \qquad (5)$$

where s is the particle spin, with fermions having $\mathsf{s} = 1/2$ and bosons $\mathsf{s} = 0$. The integrals are elementary and give

$$\begin{aligned}\Omega(T,V,\mu) &= \\ &\delta_{a,-1}\Omega_0 + \frac{2\mathsf{s}+1}{a}\beta^{-[(d+\delta)/2+1]}\left(\frac{mL^2}{2\pi\hbar^2}\right)^{(d-\delta)/2} \\ &\times(\hbar\omega)^{-\delta}\sum_{l=1}^{\infty}\frac{\left[-a\mathrm{e}^{\beta(\mu-\delta\hbar\omega/2)}\right]^l}{l^{(d+\delta)/2+1}}\cdot\end{aligned} \qquad (6)$$

The infinite sum is expressible in terms of the *polylogarithm function* $Li_\sigma(z)$ (designated by $PolyLog\,[\sigma,z]$ in Ref. [39]), since

$$\begin{aligned}-aLi_\sigma(-az) &\equiv \frac{1}{\Gamma(\sigma)}\int_0^{\infty}\mathrm{d}x\frac{x^{\sigma-1}}{z^{-1}\mathrm{e}^x+a} \\ &= -\frac{1}{a}\sum_{l=1}^{\infty}\frac{(-az)^l}{l^\sigma}\cdot\end{aligned} \qquad (7)$$

For $\sigma \geq 1$ this reduces to Bose-Einstein (BE) integrals $g_\sigma(z)$ when $a = -1$ and to Fermi-Dirac (FD) integrals $f_\sigma(z)$ when $a = 1$, as defined in Appendices D and E of reference [38], and $z \equiv e^{\beta\mu}$ is the fugacity. Using (7) in (6) leaves

$$\Omega\,(T,V,\mu) = \delta_{a,-1}\Omega_0 - \frac{1}{a}\frac{A_{d+\delta}}{\beta^{(d+\delta)/2+1}} Li_{(d+\delta)/2+1}(-az_1) \tag{8}$$

where

$$A_{d+\delta} \equiv \frac{2\mathrm{s}+1}{(\hbar\omega)^\delta}\left(mL^2/2\pi\hbar^2\right)^{(d-\delta)/2} \tag{9}$$

and

$$z_1 \equiv z\mathrm{e}^{-\beta\delta\hbar\omega/2} = \mathrm{e}^{\beta(\mu-\delta\hbar\omega/2)}. \tag{10}$$

For $d \geqslant 2$ the problem with different frequencies ω in different directions will be treated elsewhere, as well as inclusion of interaction effects.

3 Thermodynamic properties

From (8) it is possible to find the thermodynamic properties for a monatomic gas using the relation

$$\mathrm{d}\Omega = -S\mathrm{d}T - P\mathrm{d}V - N\mathrm{d}\mu. \tag{11}$$

In this representation the grand potential $\Omega(T,V,\mu) = -PV$ is the fundamental relation leading to all the thermodynamic properties of the system since

$$N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V}, \qquad S = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu},$$
$$P = -\left(\frac{\partial\Omega}{\partial V}\right)_{T,\mu} = -\frac{\Omega}{V}\cdot \tag{12}$$

Next, consider only the excited states as the ground state will be treated separately for the boson gas. Using (8) and (12) the particle number is given by

$$N = -\frac{A_{d+\delta}}{a\beta^{(d+\delta)/2}} Li_{(d+\delta)/2}(-az_1), \tag{13}$$

where we used the relation

$$\left(\frac{\partial Li_{(d+\delta)/2+1}(-az_1)}{\partial\mu}\right)_{T,V} = \beta\; Li_{(d+\delta)/2}(-az_1). \tag{14}$$

The entropy follows on substituting (8) in the first equation of (12), giving

$$S/k_\mathrm{B} = -[(d+\delta)/2+1] \times \frac{A_{d+\delta}}{a\beta^{(d+\delta)/2}} Li_{(d+\delta)/2+1}(-az_1) - N\ln z_1, \tag{15}$$

where we used the number equation (13) and the relation

$$\left(\frac{\partial Li_{(d+\delta)/2+1}(-az_1)}{\partial T}\right)_{V,\mu} = \frac{1}{z_1}\left(\frac{\partial z_1}{\partial T}\right)_{V,\mu} Li_{(d+\delta)/2}(-az_1). \tag{16}$$

Thus (15) becomes

$$S/Nk_\mathrm{B} = \frac{[(d+\delta)/2+1]Li_{(d+\delta)/2+1}(-az_1)}{Li_{(d+\delta)/2}(-az_1)} - \ln z_1. \tag{17}$$

The internal energy is obtained from (see p. 159 of Ref. [38])

$$U(T,V) = -k_\mathrm{B}T^2\left[\frac{\partial}{\partial T}\left(\frac{\Omega}{k_\mathrm{B}T}\right)\right]_{V,\ z}. \tag{18}$$

Substituting (8) here we find that

$$U(T,V) = N\frac{\hbar\omega\delta}{2} - \frac{d+\delta}{2}\Omega, \tag{19}$$

and since $\Omega = -PV$ then

$$PV = \frac{2}{d+\delta}(U - N\hbar\omega\delta/2). \tag{20}$$

Using (9) and (13) the internal energy (19) can be rewritten as

$$\frac{U(T,V)}{Nk_\mathrm{B}T} = \left[\beta\frac{\hbar\omega\delta}{2} + \frac{d+\delta}{2}\frac{Li_{(d+\delta)/2+1}(-az_1)}{Li_{(d+\delta)/2}(-az_1)}\right]\cdot \tag{21}$$

The specific heat at constant volume C_V then follows from

$$C_V = \left[\frac{\partial}{\partial T}U(T,V)\right]_{N,V} \tag{22}$$

and gives

$$\frac{C_\mathrm{V}}{Nk_\mathrm{B}} = \frac{d+\delta}{2}\left[\left(\frac{d+\delta}{2}+1\right)\frac{Li_{(d+\delta)/2+1}(-az_1)}{Li_{(d+\delta)/2}(-az_1)} - \frac{d+\delta}{2}\frac{Li_{(d+\delta)/2}(-az_1)}{Li_{(d+\delta)/2-1}(-az_1)}\right] \tag{23}$$

where we have used the relation

$$\frac{1}{z_1}\left(\frac{\partial z_1}{\partial T}\right)_{N,V} = -k_\mathrm{B}\beta\frac{d+\delta}{2}\frac{Li_{(d+\delta)/2}(-az_1)}{Li_{(d+\delta)/2-1}(-az_1)} \tag{24}$$

which can be extracted from the (vanishing) derivative with respect to T of the number equation (13). Since

$$z_1 \equiv \mathrm{e}^{\beta(\mu-\delta\hbar\omega/2)} \xrightarrow[T\to\infty]{} 0,$$

equation (7) then implies that $-aLi_\sigma(-az_1) \to z_1$, with $\sigma = (d+\delta)/2-1$, $(d+\delta)/2$ or $(d+\delta)/2+1$. Then (23) reduces to

$$\frac{C_\mathrm{V}}{Nk_\mathrm{B}} \xrightarrow[T\to\infty]{} \frac{d+\delta}{2}\left[1 + a\frac{z_1}{2^{(d+\delta)/2+1}}\left(1-\frac{d+\delta}{2}\right)\right], \tag{25}$$

which for $d=\delta=3$ gives the classical Dulong-Petit law for *crystals* when $T\to\infty$ or $z_1 = 0$, while for $\delta = 0$ we obtain the classical limit for *ideal gases* of bosons or fermions. The first correction to unity in (25) for $d+\delta < 2$ is clearly negative for $a = -1$ (bosons) and positive for $a = +1$

(fermions), while for $d+\delta > 2$ it is precisely the opposite. Thus we obtain known results obtained for ideal gases for $\delta = 0$ (bosons [40], fermions [41]).

We now generalize the results obtained in references [42–44] dealing with the identity of specific heats as a function of T of ideal Bose and Fermi gases in two dimensions. This identity is obtained here more generally for $d+\delta = 2$. If both gases are at the same temperature and have the same number density $n_{\rm B} = n_{\rm F}$, where $n_{\rm B} \equiv N_{\rm B}/V$ is the Bose and $n_{\rm F} \equiv N_{\rm F}/V$ is the Fermi density, taking $d+\delta = 2$ in (13) gives

$$n_{\rm B} = \frac{A_2}{V}\frac{Li_1(z_{1\rm B})}{\beta} = -\frac{A_2}{V}\frac{Li_1(-z_{1\rm F})}{\beta} = n_{\rm F}, \qquad (26)$$

where as before V was defined just below (3), $z_{1\rm B} \equiv \mathrm{e}^{\beta(\mu_{\rm B}-\hbar\omega\delta/2)}$ and $z_{1\rm F} \equiv \mathrm{e}^{\beta(\mu_{\rm F}-\hbar\omega\delta/2)}$ are the fugacities with $\mu_{\rm B}$ and $\mu_{\rm F}$ the chemical potentials for bosons and fermions, respectively. Using Landen's relations [43] the polylogarithm functions $Li_\sigma(z)$ satisfy $Li_1(x) = -Li_1(y)$ and $Li_2(x) = -Li_2(y) - 1/2\,[Li_1(y)]^2$, where $x \to y$ satisfy the Euler transformation $y \equiv -x/(1-x)$ with x real <1. Substituting these relations in (26), we obtain

$$z_{1\rm F} = z_{1\rm B}/\left(1 - z_{1\rm B}\right). \qquad (27)$$

The energy of the Bose gas $U(T,V)_{\rm B}$ taking $a = -1$ in (21) with $d+\delta = 2$, is

$$\frac{U(T,V)_{\rm B}}{Nk_{\rm B}T} = \left[\beta\frac{\hbar\omega\delta}{2} + \frac{Li_2(z_{1\rm B})}{Li_1(z_{1\rm B})}\right]\cdot \qquad (28)$$

Substituting (27) in (28) we obtain

$$\frac{U(T,V)_{\rm B}}{N} = \left[\frac{\hbar\omega\delta}{2} + \beta^{-1}\frac{Li_2(-z_{1\rm F})}{Li_1(-z_{1\rm F})} + 1/2\beta^{-1}Li_1(-z_{1\rm F})\right]$$
$$= \left[\frac{U(T,V)_{\rm F}}{N} + 1/2\beta^{-1}Li_1(-z_{1\rm F})\right], \qquad (29)$$

where $U(T,V)_{\rm F}$ is the Fermi gas energy. Substituting (26) in (29) the last term in (29) is proportional to $n_{\rm F}$. Hence, the energies of the Bose and Fermi gases differ only by a T-independent term and so, from (22), the specific heats for boson and fermion gases precisely coincide when $d+\delta = 2$, or

$$\left[C_{\rm V}(N,T)\right]_{\rm B} = \left[C_{\rm V}(N,T)\right]_{\rm F}. \qquad (30)$$

3.1 Mapping to higher-d and equivalent mass

Using (7) and (10), equation (13) can be rewritten as

$$N = \frac{A_{d+\delta}}{\Gamma\left[(d+\delta)/2\right]}\int_0^\infty \mathrm{d}\varepsilon\frac{\varepsilon^{(d+\delta)/2-1}}{z_1^{-1}\mathrm{e}^{\beta\varepsilon}+a}$$
$$= \frac{A_{d+\delta}}{\Gamma\left(\left[d+\delta\right]/2\right)}\int_{\hbar\omega\delta/2}^\infty \mathrm{d}\varepsilon\frac{(\varepsilon-\hbar\omega\delta/2)^{(d+\delta)/2-1}}{\mathrm{e}^{\beta(\varepsilon-\mu)}+a}$$
$$\equiv \int_{\hbar\omega\delta/2}^\infty \mathrm{d}\varepsilon\mathcal{N}\left(\varepsilon\right)n\left(\varepsilon\right), \qquad (31)$$

where $n\left(\varepsilon\right) = \left[\mathrm{e}^{\beta(\varepsilon-\mu)}+a\right]^{-1}$ is the BE ($a=-1$) or FD ($a=+1$) distribution, and $\mathcal{N}\left(\varepsilon\right)$ is the density of states (DOS). Substituting $A_{d+\delta}$ from (9) into (31) we identify this *generalized* DOS $\mathcal{N}(\varepsilon)$ as

$$\mathcal{N}(\varepsilon) = (2\mathrm{s}+1)\left(\frac{2\pi\hbar}{m\omega L^2}\right)^\delta\left(\frac{mL^2}{2\pi\hbar^2}\right)^{(d+\delta)/2}$$
$$\times\frac{(\varepsilon-\hbar\omega\delta/2)^{(d+\delta)/2-1}}{\Gamma\left(\left[d+\delta\right]/2\right)}\cdot \qquad (32)$$

If $\delta = 0$ we recover the DOS for a free gas confined in a "box" of sides L

$$\mathcal{N}_0(\varepsilon) = (2\mathrm{s}+1)\left(\frac{mL^2}{2\pi\hbar^2}\right)^{d/2}\frac{\varepsilon^{d/2-1}}{\Gamma\left(d/2\right)}. \qquad (33)$$

Comparing (32) with (33) in $(d+\delta)$-dimensions

$$\mathcal{N}_0(\varepsilon) = (2\mathrm{s}+1)\left(\frac{mL^2}{2\pi\hbar^2}\right)^{(d+\delta)/2}\frac{\varepsilon^{(d+\delta)/2-1}}{\Gamma\left(\left[d+\delta\right]/2\right)}, \qquad (34)$$

we observe that except for the (negligible) zero-point energy of $\hbar\omega\delta/2$, (32) and (34) are identical if in (32) an *equivalent* particle mass m^* defined by

$$m^* = \left(h/\omega L^2\right)^{2\delta/(d+\delta)} m^{(d-\delta)/(d+\delta)} \qquad (35)$$

is introduced. Then

$$\mathcal{N}(\varepsilon) = (2\mathrm{s}+1)\left(m^*L^2/2\pi\hbar^2\right)^{(d+\delta)/2}\frac{\varepsilon^{(d+\delta)/2-1}}{\Gamma\left(\left[d+\delta\right]/2\right)}\cdot \qquad (36)$$

In general, therefore, the effect of trapping a quantum gas renormalizes the particle mass $m \to m^*$ in accordance with (35) and increases the dimensionality $d \to d+\delta$ by the number of oscillators.

3.2 Thermodynamic limit

Substituting the coefficient $A_{d+\delta}$ from (9) into (31) gives

$$N = (2\mathrm{s}+1)\left(\frac{m}{2\pi\hbar^2}\right)^{(d+\delta)/2}\left(\frac{2\pi\hbar}{m\omega}\right)^\delta$$
$$\times\frac{x_0^{-2\delta}V}{\Gamma\left(\left[d+\delta\right]/2\right)}\int_{\hbar\omega\delta/2}^\infty \mathrm{d}\varepsilon\frac{(\varepsilon-\hbar\omega\delta/2)^{(d+\delta)/2-1}}{\mathrm{e}^{\beta(\varepsilon-\mu)}+a} \qquad (37)$$

the volume V being defined just below (3). The proper thermodynamic limit then holds if $N\to\infty$, $L\to\infty$, $\omega\to 0$ while keeping the ratio $N/V = N/L^{d-\delta}x_0^{2\delta} \propto N\omega^\delta/L^{d-\delta} = $ constant. This result was obtained for $d=3$ and $\delta = 3$ in reference [20]. For a free gas, *i.e.*, $\delta = 0$, we recover the usual thermodynamic limit $N\to\infty, L\to\infty$ with $N/L^d = $ constant.

4 Trapped bosons

In this section we study a system of N noninteracting bosons in d dimensions trapped by δ ($\leq d$) mutually-perpendicular harmonic oscillators, and otherwise free in the remaining $d-\delta$ directions. Let the boson number be

$$N = N_0(T) + N_{\mathbf{k}>0}(T) \tag{38}$$

were $N_0(T) = -\left(\partial \Omega_0/\partial\mu\right)_{T,V}$ is the number of bosons in the lowest energy state, with Ω_0 defined just below (3), while $N_{\mathbf{k}>0}(T)$ is given by (13) with $a=-1$. Thus

$$N = N_0(T) + \frac{A_{d+\delta}}{\beta^{(d+\delta)/2}} g_{(d+\delta)/2}(z_1), \tag{39}$$

where from (7) we introduce the Bose function $g_\sigma(z)$ which for $z=1$ and $\sigma>1$ is identical to the Riemann Zeta function $\zeta(\sigma)$.

Since for $T>T_\mathrm{c}$, $N_0(T)$ is negligible compared with N, while for $T<T_\mathrm{c}$, $N_0(T)$ is a sizeable fraction of N, at $T=T_\mathrm{c}$, $N_0(T_\mathrm{c})\simeq 0$. The critical temperature T_c of BEC is found from the condition $N_{\mathbf{k}>0}(T_\mathrm{c}, z_1=1)\simeq N$, so that (39) leads to

$$k_\mathrm{B}T_\mathrm{c} = \left[\frac{N}{A_{d+\delta}\, g_{(d+\delta)/2}(1)}\right]^{2/(d+\delta)}. \tag{40}$$

From (39) and (40) one obtains for the condensate fraction,

$$\begin{aligned} N_0(T)/N &\equiv 1 - N_{\mathbf{k}>0}(T)/N\,(T_\mathrm{c}) \\ &= 1-(T/T_\mathrm{c})^{(d+\delta)/2}. \end{aligned} \tag{41}$$

From (7) the infinite series $g_\sigma(1)$ diverges for $\sigma\leq 1$ implying from (40) that BEC will occur with critical temperature $T_\mathrm{c}\neq 0$ if and only if $(d+\delta)/2>1$. For $\delta=0$ and $d=3$ with $n\equiv N/L^3$ (40) reduces to the familiar formula $T_\mathrm{c}\simeq 3.31\hbar^2 n^{2/3}/mk_\mathrm{B}$ of "ordinary" BEC, since $g_{3/2}(1)=\zeta(3/2)\simeq 2.612$. On the other hand, substituting $\delta=3$ and $d=3$ in (40) and (9) we recover the result obtained in [20], since $\zeta(3)\simeq 1.202$,

$$k_\mathrm{B}T_\mathrm{c} \simeq 0.94\hbar\omega N^{1/3}. \tag{42}$$

An apparent counterexample of our no-BEC result for $d=\delta=1$ is reported in [13], but can be explained by taking $d=2$ and $\delta=1$.

The specific heat follows from (23) for $a=-1$ and from $-aLi_\sigma(-az)\,[\sigma,z_1]=g_\sigma(z_1)$. We obtain for $T>T_\mathrm{c}$

$$\begin{aligned} \frac{C_\mathrm{V}}{Nk_\mathrm{B}} = \frac{d+\delta}{2}\Bigg[&\left(\frac{d+\delta}{2}+1\right)\frac{g_{(d+\delta)/2+1}(z_1)}{g_{(d+\delta)/2}(z_1)} \\ &-\frac{d+\delta}{2}\frac{g_{(d+\delta)/2}(z_1)}{g_{(d+\delta)/2-1}(z_1)}\Bigg], \end{aligned} \tag{43}$$

while for $T\leq T_\mathrm{c}$, $z_1=1$ so that it follows directly from (21) and (22) that

$$\begin{aligned} \frac{C_\mathrm{V}}{Nk_\mathrm{B}} &= \frac{d+\delta}{2}\left(\frac{d+\delta}{2}+1\right) \\ &\times (T/T_\mathrm{c})^{(d+\delta)/2}\frac{g_{(d+\delta)/2+1}(1)}{g_{(d+\delta)/2}(1)}. \end{aligned} \tag{44}$$

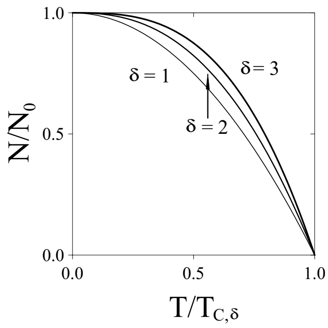

Fig. 1. Condensate fraction for a 3D boson gas trapped by $\delta = 1$, 2 or 3 harmonic oscillators.

The specific heat jump at T_c is then

$$\begin{aligned} \frac{\Delta C_\mathrm{V}}{Nk_\mathrm{B}} &= \frac{C_V(T_\mathrm{c}^-)-C_V(T_\mathrm{c}^+)}{Nk_\mathrm{B}} \\ &= \left(\frac{d+\delta}{2}\right)^2\frac{g_{(d+\delta)/2}(1)}{g_{(d+\delta)/2-1}(1)}, \end{aligned} \tag{45}$$

which is nonzero if and only if $(d+\delta)>4$, since $g_\sigma(1)$ diverges for $\sigma\leq 1$.

The entropy for the boson case $a=-1$ follows from (15). Since $-aLi_\sigma(-az)\,[\sigma,z_1]=g_\sigma(z_1)$ and using (13), in terms of the critical temperature T_c it becomes

$$\begin{aligned} S/Nk_\mathrm{B} &= [(d+\delta)/2+1] \\ &\times (T/T_\mathrm{c})^{(d+\delta)/2}\frac{g_{(d+\delta)/2+1}(z_1)}{g_{(d+\delta)/2}(1)} - \ln z_1. \end{aligned} \tag{46}$$

For $T\leq T_\mathrm{c}$, $z_1=1$, so that this becomes

$$\begin{aligned} S/Nk_\mathrm{B} &= [(d+\delta)/2+1]\,(T/T_\mathrm{c})^{(d+\delta)/2} \\ &\times \frac{g_{(d+\delta)/2+1}(1)}{g_{(d+\delta)/2}(1)} \xrightarrow[T\to 0]{} 0 \end{aligned}$$

which complies with the third law of thermodynamics.

For 3D bosons trapped by 1, 2 or 3 harmonic oscillators we summarize our results in Table 1. Since for $z_1=1$ the series $g_\sigma(z_1)$ for $\sigma>1$ coincides with $\zeta(\sigma)$, we require the following values: $\zeta(3/2)\simeq 2.612$, $\zeta(2)=\pi^2/6\simeq 1.645$, $\zeta(5/2)\simeq 1.341$, $\zeta(3)\simeq 1.202$, $\zeta(3/2)\simeq 1.127$, and $\zeta(4)=\pi^4/90\simeq 1.082$.

In Figure 1 we show the condensate fraction for $\delta=$ 1, 2 and 3. In Figure 2 shows their internal energy; specific heat at constant volume (having a jump discontinuity if and only if $d+\delta>4$); entropy and chemical potential.

An ideal Bose gas in d-dimensional space trapped by $\delta\leq d$ harmonic oscillators has its geometric dimensionality effectively reduced. The BEC temperature expression (40) for a trapped noninteracting Bose gas shows that BEC can occur if and only if $(d+\delta)/2>1$ as otherwise the term $g_{(d+\delta)/2}(1)$ diverges, forcing T_c to vanish. Thus BEC is possible in 2D provided $\delta\geq 1$.

Experiments with dilute boson gases confined in the realistic confining potentials of opto-magnetic traps in the

Table 1. Thermodynamic quantities, as defined in text, for a 3D boson gas trapped by $\delta = 3, 2, 1$ harmonic oscillators, with the oscillator length parameter $x_0 \equiv (\hbar/m\omega)^{1/2}$.

δ	3	2	1
$\mathcal{N}(\varepsilon)$	$\frac{1}{2}(\hbar\omega)^{-3}(\varepsilon-\frac{3}{2}\hbar\omega)^2$	$\frac{2^{3/2}}{3}\frac{L}{\pi x_0}(\hbar\omega)^{-5/2}(\varepsilon-\hbar\omega)^{3/2}$	$\frac{L^2}{2\pi x_0^2}(\hbar\omega)^{-2}(\varepsilon-\frac{1}{2}\hbar\omega)$
N_0/N	$1-(\frac{T}{T_c})^3$	$1-(\frac{T}{T_c})^{5/2}$	$1-(\frac{T}{T_c})^2$
T_c	$\frac{\hbar\omega}{k_B}[\frac{N}{\zeta(3)}]^{1/3}$	$\frac{\hbar\omega}{k_B}\left[\frac{(2\pi)^{1/2}N}{\zeta(3/2)}\frac{x_0}{L}\right]^{2/5}$	$\frac{\hbar\omega}{k_B}\left[\frac{2\pi N}{\zeta(2)}\frac{x_0^2}{L^2}\right]^{1/2}$
U/Nk_BT	$\frac{3}{2}\frac{\hbar\omega}{k_BT}+3(\frac{T}{T_c})^3\frac{g_4(z_1)}{\zeta(3)}$	$\frac{\hbar\omega}{k_BT}+\frac{5}{2}(\frac{T}{T_c})^{5/2}\frac{g_{7/2}(z_1)}{\zeta(5/2)}$	$\frac{1}{2}\frac{\hbar\omega}{k_BT}+2(\frac{T}{T_c})^2\frac{g_3(z_1)}{\zeta(2)}$
C_V/Nk_B $(T<T_c)$	$12\left(\frac{T}{T_c}\right)^3\frac{\zeta(4)}{\zeta(3)}$	$\frac{35}{4}\left(\frac{T}{T_c}\right)^{5/2}\frac{\zeta(7/2)}{\zeta(3/2)}$	$6\left(\frac{T}{T_c}\right)^2\frac{\zeta(3))}{\zeta(2)}$
C_V/Nk_B $(T>T_c)$	$12\frac{g_4(z_1)}{g_3(z_1)}-9\frac{g_3(z_1)}{g_2(z_1)}$	$\frac{35}{4}\frac{g_{7/2}(z_1)}{g_{5/2}(z_1)}-\frac{25}{4}\frac{g_{5/2}(z_1)}{g_{3/2}(z_1)}$	$6\frac{g_3(z_1)}{g_2(z_1)}-4\frac{g_2(z_1)}{g_1(z_1)}$
$\Delta C_V/Nk_B$	$9\frac{\zeta(3)}{\zeta(2)}\simeq 6.57$	$\frac{25}{4}\frac{\zeta(5/2)}{\zeta(3/2)}\simeq 3.20$	0
PV	$\frac{1}{3}(U-\frac{3}{2}N\hbar\omega)$	$\frac{2}{5}(U-N\hbar\omega)$	$\frac{1}{2}(U-\frac{1}{2}N\hbar\omega)$
S/Nk_B	$4\left(\frac{T}{T_c}\right)^3\frac{g_4(z_1)}{\zeta(3)}-\ln z_1$	$\frac{7}{2}\left(\frac{T}{T_c}\right)^{5/2}\frac{g_{7/2}(z_1)}{\zeta(5/2)}-\ln z_1$	$3\left(\frac{T}{T_c}\right)^2\frac{g_3(z_1)}{\zeta(2)}-\ln z_1$

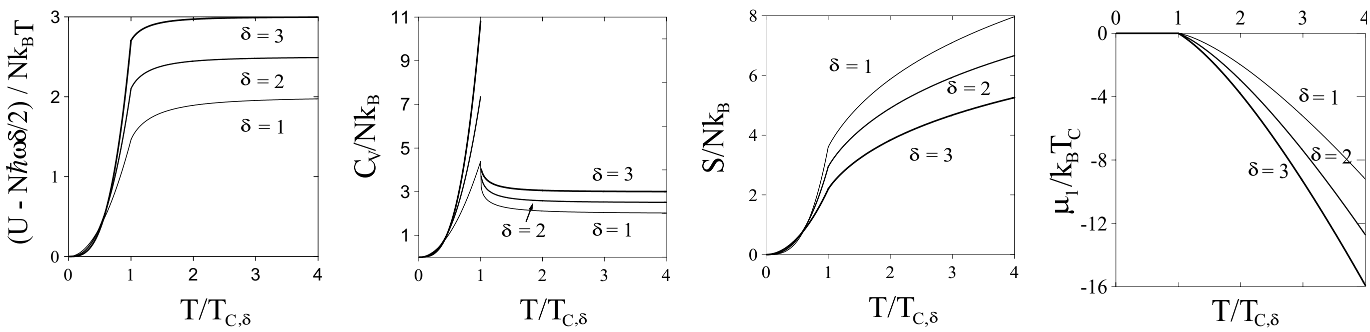

Fig. 2. Thermodynamic variables as functions of temperature T, as defined in text, for a 3D boson gas trapped by $\delta = 1$, 2 or 3 harmonic oscillators.

region of small oscillations where (BEC) has been observed, can be viewed as a Bose gas in 3D with $\delta = 3$. Table 2 shows some parameters in bosonic vapor systems where BEC has thus far been observed. The ratio in last row shows the kind of agreement obtained between the experimental T_c and calculated T_0 (42) where the good agreement found by Ensher *et al.* [45] for $^{87}_{37}$Rb is also obtained in this paper for $^{7}_{3}$Li, $^{85}_{37}$Rb, $^{4}_{2}$He and $^{41}_{19}$K. However, for $^{27}_{11}$Na and $^{1}_{1}$H we find only moderate agreement.

5 Trapped fermions

Finally, consider a system of N noninteracting fermions in d dimensions trapped by δ $(\leq d)$ mutually perpendicular harmonic oscillators, and otherwise free in the remaining $d-\delta$ directions. Since

$$\left[e^{\beta\{\varepsilon-\mu(T)\}}+1\right]^{-1} \xrightarrow[T\to 0]{} \theta\left(E_F-\varepsilon\right),$$

with $\mu(0) \equiv E_F \equiv \hbar^2k_F^2/2m$ the Fermi energy, k_F being the Fermi wavenumber, we see from (31) with $a = +1$ that

$$N \xrightarrow[T\to 0]{} 2A_{d+\delta}/\left(d+\delta\right)\Gamma\left(\left[d+\delta\right]/2\right) \times\left(E_F-\hbar\omega\delta/2\right)^{(d+\delta)/2}$$
$$\simeq\left[2A_{d+\delta}/\left(d+\delta\right)\Gamma\left(\left[d+\delta\right]/2\right)\right]E_F^{(d+\delta)/2} \quad (47)$$

where in the last step we neglected $\hbar\omega\delta/2$ compared with E_F. The fermion number density with $s = 1/2$, if $\delta = 0$ is obtained [41,46], substituting (9) in (47), as

$$n \equiv \frac{N}{L^d} = \frac{k_F^d}{2^{d-2}\pi^{d/2}d\,\Gamma(d/2)}, \quad (48)$$

which reduces to the familiar results $n = 2k_F/\pi$, $k_F^2/2\pi$ and $k_F^3/3\pi^2$ for $d = 1$, 2 and 3, respectively.

Recalling that $-Li_\sigma(-z) \equiv f_\sigma(z)$ which are the FD integrals, the internal energy from (21) can be expressed as

$$\frac{U(T,V)}{Nk_BT} = \left[\beta\frac{\hbar\omega\delta}{2}+\frac{d+\delta}{2}\frac{f_{(d+\delta)/2+1}(z_1)}{f_{(d+\delta)/2}(z_1)}\right]. \quad (49)$$

Table 2. Some experimental parameters associated with trapped bosonic gases in which BEC has been observed to date, N and N_0 being the number of atoms in the initial cloud and in the condensate, respectively; T_c the BEC transition temperature; n the boson number density, $\overline{\nu}$ an average trap frequency; and T_0 the BEC critical temperature calculated with (42) for a harmonically trapped ideal boson gas.

Boson	$^{87}_{37}$Rb	$^{27}_{11}$Na	$^{7}_{3}$Li	$^{1}_{1}$H	$^{85}_{37}$Rb	$^{4}_{2}$He	$^{41}_{19}$K
Year/Ref.	1995 [45]	1995 [2]	1995 [3]	1998 [4]	2000 [5]	2001 [6]	2001 [7]
N	4×10^4	5×10^5	2×10^5	-	-	8×10^6	-
N_c	2×10^3	-	-	10^9	10^4	5×10^5	10^4
T_c (μK)	0.28	2	0.4	50	0.015	4.7	0.16
n (cm^{-3})	-	1.5×10^{14}	2×10^{12}	4.8×10^{15}	1×10^{12}	3.8×10^{13}	6×10^{11}
$\overline{\nu}$ (Hz)	186.5	345.62	145.94	786.97	12.8	515	232.17
T_0 (μK)	0.29	1.4	0.4	36	0.012	4.6	0.12
T_c/T_0	0.97	1.4	1	1.4	1.25	1.02	1.33

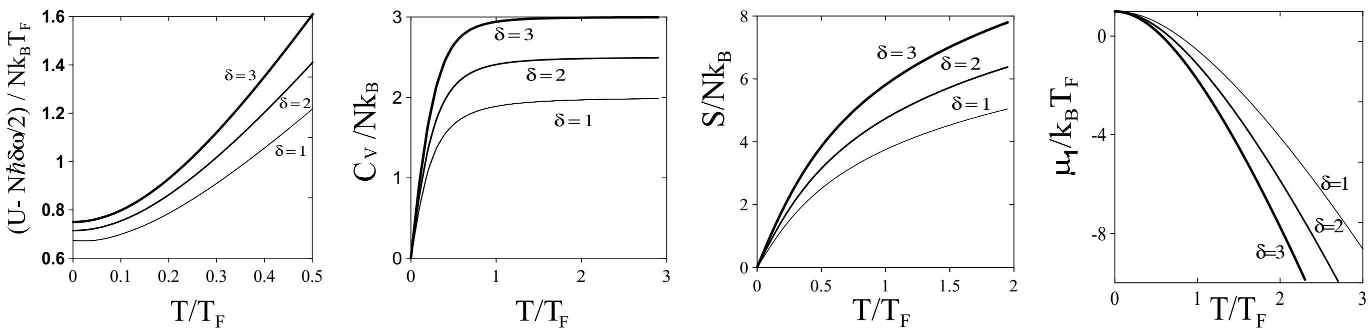

Fig. 3. Same as Figure 2 but for a fermion gas.

Table 3. Thermodynamic quantities, as defined in text, for a 3D fermion gas trapped by $\delta = 1, 2, 3$ harmonic oscillators.

δ	3	2	1
N	$\frac{2}{3}(\hbar\omega)^{-3}E_F^3$	$\frac{4}{5}\left(\frac{mL^2}{2\pi\hbar^2}\right)^{1/2}(\hbar\omega)^{-2}E_F^{5/2}$	$\left(\frac{mL^2}{2\pi\hbar^2}\right)(\hbar\omega)^{-1}E_F^2$
U/Nk_BT	$\frac{3\hbar\omega}{2k_BT}+\frac{3f_4(z_1)}{f_3(z_1)}$	$\frac{\hbar\omega}{k_BT}+\frac{5f_{7/2}(z_1)}{2f_{3/2}(z_1)}$	$\frac{\hbar\omega}{2k_BT}+\frac{2f_3(z_1)}{f_2(z_1)}$
C_V/Nk_B	$12\frac{f_4(z_1)}{f_3(z_1)}-9\frac{f_3(z_1)}{f_2(z_1)}$	$\frac{35}{4}\frac{f_{7/2}(z_1)}{f_{5/2}(z_1)}-\frac{25}{4}\frac{f_{5/2}(z_1)}{f_{3/2}(z_1)}$	$6\frac{f_3(z_1)}{f_2(z_1)}-4\frac{f_2(z_1)}{f_1(z_1)}$
PV/Nk_BT	$\frac{f_4(z_1)}{f_3(z_1)}$	$\frac{f_{7/2}(z_1)}{f_{5/2}(z_1)}$	$\frac{f_3(z_1)}{f_2(z_1)}$
S/Nk_B	$\frac{4f_4(z_1)}{f_3(z_1)}-\ln z_1$	$\frac{7f_{7/2}(z_1)}{2f_{5/2}(z_1)}-\ln z_1$	$\frac{f_3(z_1)}{f_2(z_1)}-\ln z_1$

Using (47) and the asymptotic expansion for $f_{d/2}(z)$ for $T \to 0$ (Ref. [41], Appendix B), (49) becomes

$$\frac{U(T)-N\hbar\omega\delta/2}{Nk_BT_F}\xrightarrow[T\to0]{}\frac{(d+\delta)}{(d+\delta+2)}\left[1+(d+\delta+2)\frac{\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]\cdot \quad (50)$$

From (23) the specific heat as $T \to 0$ is then

$$\frac{C_V(T)}{Nk_B}\xrightarrow[T\to0]{}(d+\delta)\frac{\pi^2}{6}\left(\frac{T}{T_F}\right)\cdot \quad (51)$$

Note that this is identical for $(d+\delta) = 2$ with the $T \to 0$ limit in the Bose expression (43), since there vanishing T_c makes $\mu_1 \equiv (\mu - \hbar\omega\delta/2) \to 0$, so that $z_1 \to 1$, while $g_0(1) \to \infty$, and (39) was used along with (47) for $n_B = n_F$. Clearly, this is a special case of the general result (30). In the same $T \to 0$ limit the fermion entropy $S = \int_0^T dT' C_V(T')/T'$ using (51) immediately becomes

$$S/Nk_B \xrightarrow[T\to0]{}(d+\delta)\frac{\pi^2}{6}\left(\frac{T}{T_F}\right) \quad (52)$$

which again is in agreement with the third law of thermodynamics. These results are displayed in Figure 3. Table 3 summarizes results for 3D fermions with $\delta = 1, 2, 3$ harmonic oscillators.

6 Conclusions

After constructing the grand potential, thermodynamic properties were determined along with the densities of states for ideal boson and fermion gases in d dimensions trapped by δ mutually perpendicular harmonic oscillators (HO). Trapping *maps* the system into a free gas with a new dimensionality increased by the number of trapping oscillators, specifically, $d \to d+\delta$, and renormalizes the particle masses $m \to m^*$ according to (35). In particular, we detailed how 3D boson and fermion gases trapped by 1, 2 or 3 mutually-perpendicular HO wells map into a free gas in 4, 5 and 6 dimensions, respectively. Also, it was found that in a trapped boson gas Bose-Einstein condensation with critical temperature $T_c \neq 0$ occurs if and only if $d+\delta > 2$ so that for $\delta \geq 1$, d need not be restricted to $d > 2$. Finally, for $d = \delta = 3$ the critical temperature formula reproduces the experimental values of $^{87}_{37}$Rb $^{7}_{3}$Li, $^{85}_{37}$Rb, $^{4}_{2}$He and $^{41}_{19}$K quite well, but only moderately so for $^{27}_{11}$Na and $^{1}_{1}$H.

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