Review

International Journal of Modern Physics B
Vol. 21, No. 21 (2007) 3657–3686

World Scientific
www.worldscientific.com

GENERALIZED COOPER PAIRING IN SUPERCONDUCTORS

M. DE LLANO

Instituto de Investigaciones en Materiales,
Universidad Nacional Autónoma de México,
Apdo. Postal 70-360, 04510 México, DF, Mexico
dellano@servidor.unam.mx

J. F. ANNETT

H.H. Wills Physics Laboratory, University of Bristol,
Bristol BS8 1TL, UK

Received 26 June 2007

We review Cooper pairing starting from its simplest, original 1956 version of two electrons interacting above the Fermi sea of an ideal Fermi gas (IFG). The two-electron interaction assumed extensively (if not exclusively), is the attractive two-parameter Cooper, and then BCS, model interactions. Hole Cooper pairs (CPs) and electron-hole CPs are then included along with the initial electron-CPs in terms of the single-fermion Green functions implied by the Bethe-Salpeter (BS) integral equation in the ladder approximation. A purely-imaginary CP energy "instability" is recovered that is well-documented in the literature at least since the late 1950's. A novel interpretation of this instability is that an unperturbed Hamiltonian different from the IFG one first used by Cooper suffices to obtain meaningful CPs. Instead of the IFG sea, a BCS-correlated Fermi "sea" used in the BS equation interpreted as the associated unperturbed Hamiltonian leads to real CP energies (with small imaginary terms implying damping). We survey how this has been achieved in 1D, 2D and 3D, and give a more detailed treatment in 2D. A vital distinction is that the original and generalized CPs are true bosons in contrast with BCS pairs that are not ordinary bosons but rather "hard-core bosons" as they do not obey strict Bose commutation rules. Another important common element of the original or generalized CPs (particularly in 2D where ordinary Bose-Einstein condensation (BEC) does not occur) is their *linear* dispersion relation in leading order in the total (or, center-of-mass) momentum power-series expansion of the CP energy. This theory encompasses, in principle, all empirically known superconductors including quasi-2D superconductors such as cuprates and the ET organic compounds, as well as quasi-1D ones such as the organometallic Bechgaard salts and nanotubes.

Keywords: Cooper electron- and hole-pairs; BCS-correlated ground state; Bose-Einstein condensation.

1. Introduction

Now fifty years old, the 1957 Bardeen Cooper and Schrieffer (BCS) theory of superconductivity[1] is rightly regarded as one of the most outstanding achievements

of theoretical many-body physics. The central concept of the theory is the idea of a Cooper pair. In the original model of Cooper, this was simply a two-electron bound state, relative to a full Fermi sea.[2] In the full BCS theory, this original concept was extended to a full many-body ground state, in which all electrons share in the general "pairing correlations." The theory not only provided a microscopic model for superconductivity, but it also made many highly specific and quantitative predictions including explaining the isotope effect, predicting the $T = 0$ energy gap Δ obeying the universal relation, $2\Delta = 3.53\ k_B T_c$, and in explaining temperature dependences of ultrasonic attenuation and NMR relaxation rates.[3]

For many decades, the theory, including its extensions into the strong-coupling regime, appeared to be capable of explaining all of the then known superconducting elements and compounds. This situation continued while the highest T_c value for any superconductor (SC) was 23 K, until the discovery[4] in 1986 of the first so-called "high-T_c" cuprate SC $La_{2-x}Ba_xCuO_4$ having a $T_c \simeq 35$ K. The discovery of superconductivity at 92 K in $YBa_2Cu_3O_{7-\delta}$[5] was followed by a search for materials with even higher T_c's and led, within just seven years to the highest-T_c superconductor known to date, the HgBaCaCuO cuprate[6] with a $T_c \simeq 164$ K under very high pressure ($\simeq 310,000$ atm).

Just over twenty years after the discovery of high-temperature superconductivity in cuprate materials, it is clear that many important questions still remain to be answered. As well as the still unresolved problem of the pairing dynamical mechanism and many-body excitations in the normal state of the high T_c cuprate materials, there are now also many other recently discovered materials where it is unlikely that BCS theory is applicable, at least in its original form. These include oxide materials (such as the cubic bismuthate $Ba_{1-x}K_xBiO_3$), borides (such as MgB_2), borocarbides (e.g., YNi_2B_2C), carbon-based materials (including fullerides, nanotubes, intercalated graphite, and organic conductors), and new high pressure phases of elements[7] (such as Fe, S and Ca) and simple binary and ternary compounds. All of these classes of materials have shown superconductivity above 10 K, including several up to nearly 40 K. Superconductivity at up to 84 K has even been reported in a cubic ruthenate.[8–10]

The ongoing debate about the pairing mechanism in cuprate high T_c materials has broadly led to two main schools of thought. On the one hand, P. W. Anderson argued from the very beginning[11] that cuprate materials are in a completely different class from other superconducting materials, and as such, they must have a completely new pairing mechanism quite different from the BCS theory. In addition to his original "resonating valence bond" (RVB) model, a large range of theories have focussed on superconductivity driven chiefly by *repulsive* interactions dominated by the on-site Coulomb-repulsion Hubbard U. These include gauge theories,[12] spin-fluctuation theories,[13–15] and the "Gossammer superconductivity" picture of Laughlin.[16–18] The discovery of a $d_{x^2-y^2}$ symmetry order parameter[19,20] is generally consistent with pairing mechanisms derived from a large positive U, and there is some numerical evidence for a $d_{x^2-y^2}$ symmetry ground state in the two-dimensional

square lattice Hubbard model.[21] However, it remains unclear whether the positive U Hubbard model alone can describe the hugely complex normal and superconducting state phenomenology of the cuprate materials[22] including the characteristic doping dependences, pseudogaps, marginal Fermi liquid normal state, isotope effects, and lattice inhomogeneities such as stripes.

On the other hand, many others have taken the view that it is not a completely new theory that is needed, but rather one that should extend or generalize BCS theory to describe these new materials. This approach has the advantage of building upon the foundations of BCS, and furthermore, does not necessarily imply that cuprate superconductivity is in a completely new class of materials. Rather, they may be related to other materials but just in a new parameter regime where the usual approximations of BCS (even including Eliashberg strong-coupling corrections) may not be adequate. Some of the many theoretical models which have been examined in this context include: boson-fermion models,[23–28] bipolarons,[29] the "pre-formed pair" or BCS-BEC crossover scenario,[30–34] non-adiabatic superconductivity,[35] and generalized Bose-Einstein condensation of Cooper pairs.[36–38]

The purpose of this review is to survey some of the underlying assumptions of the original BCS theory, especially focussing on the role and meaning of the concept of a Cooper pair (CP). We examine both the original Cooper problem, the Cooper instability of the normal Fermi sea, and the meaning of a CP *within* the BCS ground state. We recall some old debates, such as whether Cooper pairing is a form of BEC, and whether it is meaningful to talk about "pre-formed" CPs at temperatures above T_c. We examine CPs with finite center-of-mass momentum, and demonstrate that such CPs have linear dispersion, as well as a finite resonance lifetime. We draw a distinction between these two-body bound CPs, and the pair-like correlations existing in the BCS ground state wave function. In particular, we discuss the implications of pair-pair interactions, which are neglected in the BCS theory, and we discuss the difference between two-electron (charge $-2e$) and two hole ($+2e$) CPs, which is also ignored in the BCS theory, at least explicitly. Finally we briefly summarize how these ideas lead to the derivation of boson-fermion models as extensions of BCS theory, and the concept of generalized BEC (GBEC)[36–38] in which both hole- and electron-pairs are treated fully on an equal footing.

Boson-fermion (BF) models as a BEC "paradigm"[39–41] of superconductivity go back to the mid-1950's,[42–48] pre-dating even the BCS-Bogoliubov theory.[1,49,51] Although BCS theory only contemplates the presence of "Cooper correlations" of single-particle states, BF models[42–48,52–54] posit the existence of actual *bosonic* CPs. Such paired charge carriers have been observed in magnetic flux quantization experiments on elemental[55,56] as well as on cuprate[57] superconductors (SCs). Clusters larger than pairs, viz., *quartets* or quadruples with charge $\pm 4e$, have *not* been unambiguously observed in the bulk of any superconductor (see, however Refs. 58–62). The presence of quartets has also been suggested in ^{3}He in aerogel.[63] Moreover, no experiment has been done yet, to our knowledge,[64] that distinguishes between electron and hole pairs, i.e., that determines the *sign* of $2e$ charge carriers.

Indeed, CPs appear to be the single most important universally accepted ingredient of SCs, whether conventional or "exotic," and whether of low- or high-transition-temperatures T_c. And yet, in spite of their centrality, they are poorly understood. The fundamental drawback of early[42–46–48] BF models, which took $2e$ pairs as analogous to diatomic molecules in a classical atom-molecule binary gas mixture, is the cumbersome extraction of an electron energy gap $\Delta(T)$. "Gapless" models can be useful in locating transition temperatures if approached from above, i.e., $T > T_c$. Even so, we are not aware of any calculations with the early BF models attempting to reproduce any empirical T_c values. The gap first began to appear naturally in later BF models.[27–34,36–38] With two[36,37] exceptions, however, all BF models neglect the effect of *hole* CPs included on an equal footing with electron CPs to give the "complete" BF model that constitutes a generalized Bose–Einstein condensation (GBEC) theory. It is complete only in that it consists of *both* bosonic CP species coexisting with unpaired electrons.

By reviewing the CP concept, and its generalizations, we aim to clarify some commonly held, but at best questionable accepted notions regarding CPs that are widespread in conventional wisdom. In particular, we argue that the original Cooper concept[2] can be generalized to describe individual pairs relative, not to a normal Fermi sea, but relative to the correlated BCS ground state. These generalized CPs are then elementary excitations relative to the BCS ground state. They are long-lived charge $-2e$ (or $+2e$) quasiparticles of electron (or hole) pairs excited out of the BCS condensate having a characteristic linear dispersion relation and resonance lifetime, and which are described by boson statistics. We examine the nature of these generalized CP excitations, and their implications for theories of superconductivity beyond BCS.

2. Questionable Notions of Cooper Pairs (CPs)

The widely accepted notions of CPs that are at best questionable can be summarized in the following assertions stating that:

1. CPs have "a very strong preference for singlet, zero-momentum pairs, so strong that one can get an adequate description of superconductivity by treating these correlations alone."[65] In fact, the possible number of nonzero-momentum pairs has been shown[66] to be substantial. They are essential in distinguishing, within a BEC scenario, a "depairing" or "pseudogap"[67–69] temperature T^* from the critical condensation temperature T_c, which in BCS theory are otherwise the same.

2. CPs are negative-energy stable (i.e., stationary or infinite-lifetime) bound states[2] given by

$$\mathcal{E}_0 = -2\hbar\omega_D/(e^{2/\lambda} - 1) \xrightarrow[\lambda\to 0]{} -2\hbar\omega_D e^{-2/\lambda} \tag{1}$$

where $\lambda \equiv N(E_F)V$ is a dimensionless coupling constant and $N(E_F)$ the electronic density of states (DOS) for one spin evaluated at the Fermi surface of energy E_F. In fact, a more complete study of the two-body propagator in a (including both 2e-

and 2h-CPs), leads to the purely imaginary result

$$\mathcal{E}_0 = \pm i2\hbar\omega_D/\sqrt{e^{2/\lambda}-1} \underset{\lambda\to 0}{\longrightarrow} \pm i2\hbar\omega_D e^{-1/\lambda} \tag{2}$$

as reported in Ref. 70, p. 33 and Ref. 3, p. 168. Thus the original Cooper problem[2] describes an instability of the normal state, rather than merely the existence of stable two-body bound states as given by Eq. (1).

3. CPs have effective mass $2m$ where m is the single-electron effective mass. If correct, this would imply that a CP propagates in the Fermi sea like $\hbar^2K^2/4m$, with $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$, the total or center-of-mass momentum (CMM) of the pair. This is the correct energy-momentum (dispersion) relation for a composite particle of mass $2m$ *in vacuo*, but it is not correct when propagation takes place in the Fermi sea. The effective mass $2m$ describes the stiffness of the condensate wave function, as described in the Ginzburg-Landau equations, but this is a collective mode of the whole condensate, and not the dispersion of an individual CP.

4. CPs have a dispersion relation in 3D given by

$$\mathcal{E}_K - \mathcal{E}_0 = \frac{1}{2}v_F\hbar K \tag{3}$$

reported, e.g., in Ref. 3, p. 33, but is claimed to be just the sound mode with speed $v_F/\sqrt{3}$ in an ideal Fermi gas, where $v_F \equiv \sqrt{2E_F/m}$. In fact, there are *two distinct* excitation modes in a many-fermion system (that in a many-boson system are identical[71,72]), a "sound" as well as a composite "particle" mode. Below, we shall refer to the latter as a "moving-CP" to distinguish it from the sound mode.

5. CPs and BCS pairs are the same thing. By BCS pairs here we mean the $\mathbf{K} = 0$ states created by the pair operators $\hat{a}^\dagger_{\mathbf{K}/2+\mathbf{k}\uparrow}\hat{a}^\dagger_{\mathbf{K}/\mathbf{2}-\mathbf{k}\downarrow}$ where $\hat{a}^\dagger_{\mathbf{k}_i\sigma_i}$ are the creation operators for the ith electron, $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$ and $\mathbf{k} \equiv 1/2(\mathbf{k}_1 - \mathbf{k}_2)$. The full BCS ground state variational wave function is constructed as a coherent state of these $\mathbf{K} = 0$ pair operators. CPs, as we shall define precisely below, are defined as a physical elementary excitation relative to the BCS ground state, not by the BCS pair operators.

6. CPs are *not* bosons, Ref. 3, p. 38. While the BCS pair operators do not obey strict Bose commutation relations, the CP excitations which we describe below are two-particle resonances, and they do obey Bose statistics,[73] equivalent to the ordinary CPs.

7. "CPs have a very large size ($\gg$ interparticle spacing) and cannot be approximated by bosons." In fact, according to a 1931 theorem by Ehrenfest and Oppenheimer,[74] two identical clusters of charges obey the same statistics as in previous cases when infinitely separated, compared to when they are very close to each other, provided that they do not mutually excite their internal structures. We argue that this is the case for the generalized CP which we describe below, and hence they are true bosons independent of their spatial extent.

In all of these issues, there is sometimes a confusion of terminology, the phrase CP being used for different objects in different contexts by different authors. There

is also some confusion about the very different physics of the original Cooper problem, the construction of the BCS ground state, and of the excitations relative to that ground state. By the term generalized CP, or simply CP, we try here to impose a uniform nomenclature in which the true physical picture is clarified. The CPs we discuss are neither the original Cooper two-electron bound states nor the BCS pair operator states. In contrast, our choice of the term CP refers to a true physical excitation in SCs. In the remainder of this review, we expand and explain this view in greater detail. We begin by first reviewing the original CP concept, before describing its most natural generalization as a true physical CP object.

3. Original Cooper Pairing

We first review the original CP concept as a two-body bound state relative to an ideal Fermi sea. We show that linear dispersion arises in both the "ordinary" CP problem,[2] as well as in the "generalized" CP case of the next section.

The CP equation for the energy $\mathcal{E}_K$ of two fermions above the Fermi surface with momentum wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ (and arbitrary CMM wavenumber K where $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$) is given by

$$[\hbar^2 k^2/m - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m]\psi_{\mathbf{k}} = -\sum_{\mathbf{q}}{}' V_{\mathbf{kq}}\psi_{\mathbf{q}} \tag{4}$$

where $\mathbf{k} \equiv 1/2(\mathbf{k}_1 - \mathbf{k}_2)$ is the CP relative momentum and $\psi_{\mathbf{k}}$ is its wave function in momentum space. The prime on the summation implies restriction to states *above* the Fermi surface with energy $E_F \equiv \hbar^2 k_F^2/2m$, viz., $|\mathbf{k} \pm \mathbf{K}/2| > k_F$, and $V_{\mathbf{kq}}$ is the double Fourier transform of the interaction defined as

$$V_{\mathbf{kq}} \equiv \frac{1}{L^d}\int d\mathbf{r} \int d\mathbf{r}' e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}, \mathbf{r}') e^{i\mathbf{k}\cdot\mathbf{r}'} \tag{5}$$

with $V(\mathbf{r}, \mathbf{r}')$ the (possibly nonlocal) interaction in real d-dimensional space.

The number N_K of *possible* pairs with a given K is proportional to the probability of finding one electron with wavevector $\mathbf{k}_1$ and a second electron with wavevector $\mathbf{k}_2$, which combine to give a resultant CMM wavenumber K. At $T = 0$, this probability is simply the volume V_K in k-space, as this consists of a simple-cubic lattice (of lattice spacing $2\pi/L$ with L the size of the box containing the system) of points, each of which represents an electron state with either spin. Specifically

$$N_K \propto \int' d\mathbf{k}(1 - n_{k_1})(1 - n_{k_2}) \xrightarrow[T\to 0]{} V_K = \int' d\mathbf{k}\, \theta(k_1 - k_F)\theta(k_2 - k_F) \tag{6}$$

where n_{k_i} is the average number of electrons with wavevector of magnitude k_i, $i = 1, 2$, and where the prime means integration over only the overlapping volume allowed by the Cooper model interaction, written as

$$V^K_{\mathbf{k},\mathbf{k}'} = \begin{cases} -V\,\theta(2\sqrt{k_F^2 + k_D^2} - K) & \text{if } \; k_F < \left|\mathbf{k} \pm \frac{1}{2}\mathbf{K}\right|, \; \left|\mathbf{k}' \pm \frac{1}{2}\mathbf{K}\right| < \sqrt{k_F^2 + k_D^2} \\ 0 & \text{otherwise}\,. \end{cases} \tag{7}$$

As in the original CP problem,[2] we restrict ourselves to $T = 0$. In this way, the magnitudes of the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ must be inside the interval $[\,k_F,\ k_F + \hbar\omega_D]$ which form shells of energy thickness $\hbar\omega_D$ about the Fermi energy. Actually, bound pairs will be formed only if they are subject to the attractive interaction (7), i.e., only if the tip of their relative momentum $\mathbf{k}$ is within the overlap volume of both shells, the center-to-center distance of these spherical shells being equal to the magnitude of $\mathbf{K}$.

To calculate the number of possible pairs with $K > 0$ relative to those with $K = 0$, we combine conditions (7) into the primed integral (6), which then becomes the overlap volume in k-space

$$V_K \equiv \int d\mathbf{k}\,\theta(|\mathbf{K}/2 + \mathbf{k}| - k_F)\theta(|\mathbf{K}/2 - \mathbf{k}| - k_F)$$
$$\times\,\theta\Big(\sqrt{k_F^2 + k_D^2} - |\mathbf{K}/2 + \mathbf{k}|\Big)\theta\Big(\sqrt{k_F^2 + k_D^2} - |\mathbf{K}/2 - \mathbf{k}|\Big)\,. \qquad (8)$$

Though tedious, this is a straightforward multiple integration, and yields the overlapping volume of the two shells of thickness $\hbar\omega_D \equiv \hbar^2 k_D^2/2m$. Note that the overlap volume of two *spheres* is well-known (Ref. 75, p. 28) and corresponds to the particular case $k_D^2/k_F^2 \to \infty$ above. As the number of pairs for any $K \geq 0$ is proportional to the overlap volume of the corresponding shells, the ratio required is just V_K/V_0. There are five distinct topologies associated with the overlap volumes generated by separating the two perfectly coincident shells, depending on the magnitude K of vector $\mathbf{K}$, which is the center-to-center separation of the shell centers, Fig. 1. These topologies are displayed in Fig. 2, with shaded (overlap) areas designated as follows: (i) *Spherical-shell*; (ii) *Non-spherical-shell*; (iii) *Annular-ring*; (iv) *Dimpled-double-convex-lens shape*; and (v) *Double-convex-lens shape*. When the CMM wavenumber K of the pair is zero, or $\kappa \equiv K/(2\sqrt{k_F^2 + k_D^2}) = 0$, with $k_D^2/k_F^2 \equiv \hbar\omega_D/E_F \equiv \nu$,

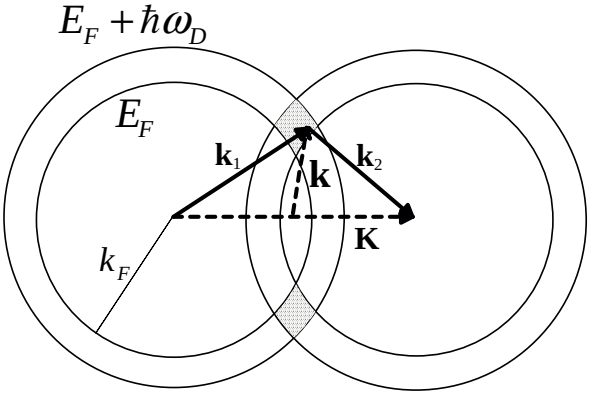

Fig. 1. Cross-section (shading) of overlap volume in 3D (area in 2D) in k-space where the tip of the relative CP wavevector $\mathbf{k} \equiv 1/2(\mathbf{k}_1 - \mathbf{k}_2)$ must point for the attractive Cooper model interaction (7) to be nonzero for a CP of fixed CMM $\hbar\mathbf{K} \equiv \hbar(\mathbf{k}_1 + \mathbf{k}_2)$.

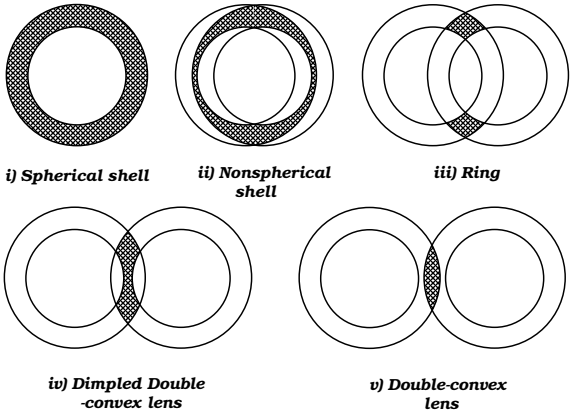

Fig. 2. Distinct topologies associated with the overlapping volume of two spherical shells in k-space as the magnitude of $\mathbf{K}$, their center-to-center distance, is increased.

the wavevectors of the individual electrons satisfy $\mathbf{k}_1 = -\mathbf{k}_2$ so that the electrons of the pair lie in a spherical shell of internal radius k_F and external radius $\sqrt{k_F^2 + k_D^2}$, at the same distance from the center and diametrically opposite each other. The centers of the two spherical shells of width $\hbar\omega_D$ are fixed at the origin, and the overlap volume defined by integral (8) is just

$$V_0 \equiv V_i = (4\pi/3)k_F^3[(1+\nu)^{\frac{3}{2}} - 1]\,. \tag{9}$$

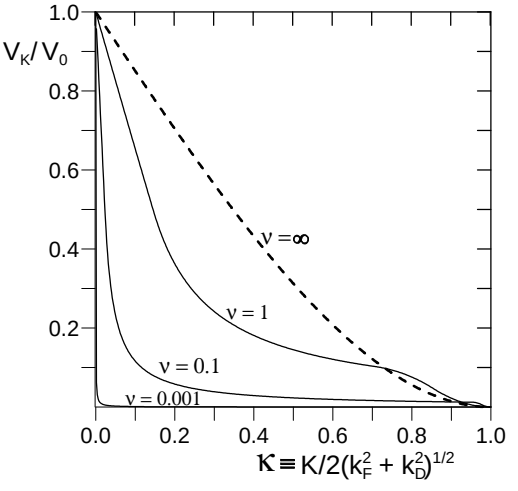

Fig. 3. Fraction of possible bound pairs with nonzero CMM, $\hbar K > 0$, to that of possible bound pairs with zero CMM, in 3D, for several values of $\nu = k_D^2/k_F^2$. Typically, $\nu = 0.005$[148] for conventional 3D, while $0.05 \leq \nu \leq 0.14$ for quasi-2D cuprate,[149] superconductors. The limit $\nu = \infty$ refers to the well-known (Ref. 75 p. 28) overlap volume in k-space of two solid spheres of radii k_F and center-to-center distance equal to K, and analytically recovered correctly in Ref. 66 where further details of Figs. 2–5 can be found.

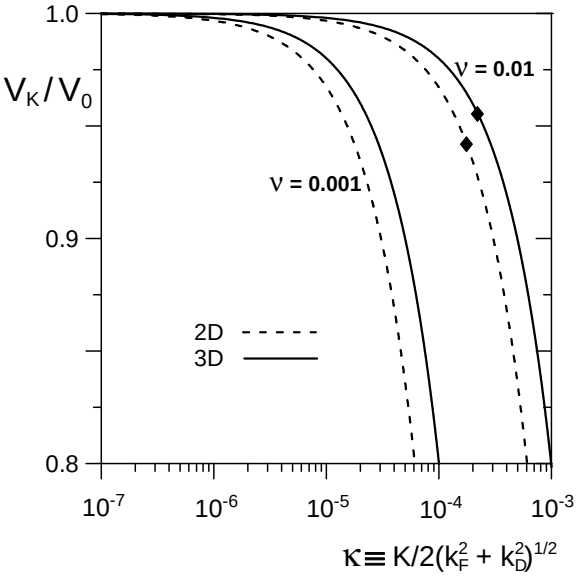

Fig. 4. Enlargement of Fig. 3 over the interval $10^{-7} \leq \kappa \equiv K/2\sqrt{k_F^2 + k_D^2} \leq 10^{-3}$, except that dashed curves here refer to 2D results. Diamond symbols mark the Cooper-pair breakup total momenta values in the linear approximation of Ref. 90.

On the other hand, when $K > 0$, the tip of wavevector $\mathbf{k}_1$ must lie, say, in the shell centered at the left, while the tail of wavevector $\mathbf{k}_2$ must lie in the shell centered at the right, both shell centers being separated by a distance K. As K is increased from zero, the overlap volume of the two shells then acquires the four shapes mentioned above. These four distinct cases are labeled (ii) through (v). Figures 3 and 4 are *not* consistent with the first accepted notion associated with CPs mentioned above.

Finite-CMM Cooper pairing also emerges in the Fulde-Ferrell-Larkin-Ovchinnikov[76,77] superconducting phase. Though not yet convincingly observed in any conventional superconductor, there is some hope[77,78] that it might be seen[80] in trapped atomic fermion gases.

3.1. *Delta-potential interaction between electrons*

If the interfermion interaction $V(\mathbf{r}, \mathbf{r}')$ is local, then $V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')$ in Eq. (5). Moreover, if $V(\mathbf{r}) = -v_0\delta(\mathbf{r})$ with $v_0 > 0$, Eq. (5) gives $V_{\mathbf{kq}} = -v_0/L^d$ and Eq. (4) becomes, for any d,

$$\frac{1}{L^d}\sum_{\mathbf{k}}{}' \frac{1}{\hbar^2 k^2/m - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m} = \frac{1}{v_0}\,. \tag{10}$$

An attractive δ-interaction can model an interelectron short-ranged attraction resulting from a short-range Coulomb repulsion plus a longer-ranged phonon attraction. In $d = 1$ where the δ-well supports a single bound state, the problem is quite

tractable.[81,82] In either 2D or 3D, however, the δ-well supports an infinite set of bound levels with the lowest level in each case being infinitely bound. This in turn leads to a rigorous collapse of the many-fermion system,[83] at least in 3D. To prevent this unphysical collapse, the 3D δ-wells must be "regularized," i.e., constructed, say, from square wells[84] such that the remaining δ-well possesses only one bound level. This leaves an infinitesimally small strength parameter v_0 which would make the rhs of Eq. (10) diverge (so as to cancel the lhs that also diverges in 2D and 3D but not in 1D). Combining Eq. (10) for 2D with the vacuum two-body Schrödinger equation in momentum representation for the same δ-potential well allows eliminating[85] v_0 in favor of the (positive) binding energy B_2 of the single bound level of the regularized δ-well. It is this same interaction that allowed Miyake[86] to explicitly solve in closed form for the zero-temperature gap $\Delta = \sqrt{2B_2E_F}$ and the fermion chemical potential $\mu = E_F - 1/2B_2$ in a 2D many-fermion gas; clearly both Δ and μ depend explicitly on both coupling through B_2 and electron number density through E_F. For the same δ-potential-well interaction in the CP problem, one arrives at

$$\sum_{\mathbf{k}} \frac{1}{B_2 + \hbar^2k^2/m} = \sum_{\mathbf{k}}{}' \frac{1}{\hbar^2k^2/m - 2E_F - \mathcal{E}_K + \hbar^2K^2/4m} \tag{11}$$

where $B_2 \geq 0$ serves as a coupling constant as in the Miyake BCS problem. A small-K power-series expansion for $\mathcal{E}_K$ gives the analytic expression valid for *any* dimensionless coupling $B_2/E_F \geq 0$,

$$\varepsilon_K \equiv \mathcal{E}_K - \mathcal{E}_0 = \frac{2}{\pi}\hbar v_F K + \left[1 - (2 - [4/\pi]^2)\frac{E_F}{B_2}\right] \frac{\hbar^2K^2}{2(2m)} + O(K^3) \tag{12}$$

where a nonnegative *CP excitation energy* ε_K has been defined, and the Fermi velocity v_F comes from $E_F/k_F = \hbar v_F/2$. The leading term in Eq. (12) is linear in K, followed by a quadratic term. It is clear that the leading term in Eq. (12) is quadratic, namely

$$\varepsilon_K = \hbar^2K^2/2(2m) + O(K^3) \tag{13}$$

provided v_F and hence E_F vanish, i.e., there is no Fermi sea associated with the pair, for *any* fixed coupling $B_2 > 0$. This is just the familiar nonrelativistic kinetic energy in vacuum of the composite (so-called "local") pair of mass $2m$ and CMM K. The same result (13) is also found to hold in 3D, but not analytically as in 2D. An analytic result with both linear and quadratic terms such as Eq. (12) was also found[87] for the attractive δ-potential-well interaction in 1D, and the same vacuum limit (13) ensued for both v_F and E_F taken to zero.

Figure 5 shows exact numerical results (full curves) of a dimensionless CP excitation energy $\varepsilon_K/(-\mathcal{E}_0)$ (in the figure, Δ_0 means present $-\mathcal{E}_0 > 0$) as a function of K/k_F for different couplings B_2 in units of E_F. Note that ordinary CPs *break up* whenever $\mathcal{E}_K$ turns from negative to positive, i.e., when $\mathcal{E}_K$ vanishes, or by Eq. (12) when $\varepsilon_K/(-\mathcal{E}_0) \equiv \varepsilon_K/\Delta_0 = 1$. These points are marked in the figure by dots on the upper abscissa. In addition to the exact results (full curves), also shown are some

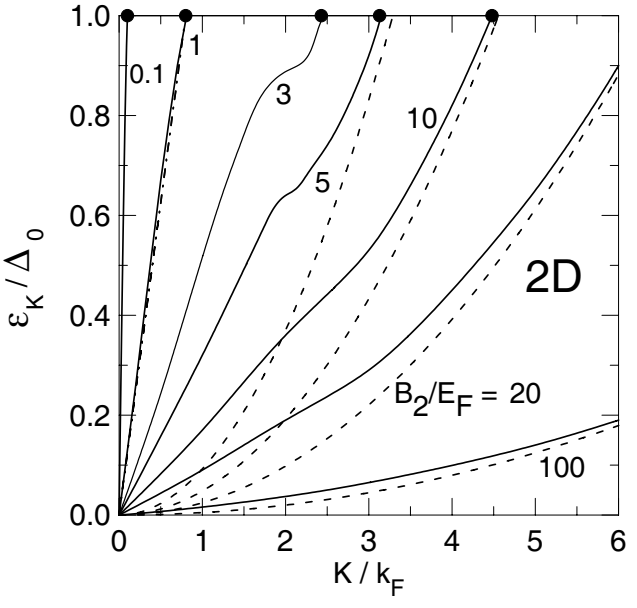

Fig. 5. Linear to quadratic changeover of dimensionless ordinary-CP excitation energy $\varepsilon_K/(-\mathcal{E}_0) \equiv \varepsilon_K/\Delta_0$ versus K/k_F, determined exactly by numerical methods from Eq. (11) for different couplings B_2/E_F, full curves, for the delta-potential-well interfermion interaction in 2D. The dot-dashed line associated with $B_2/E_F = 1$ is the linear approximation (virtually coincident with the exact curve for $B_2/E_F \lesssim 0.1$) while the dashed curve is the parabolic pure quadratic term of Eq. (12). Dots on the upper abscissa denote values of CMM wavenumber K where the CP breaks up, i.e., for which $\mathcal{E}_K \equiv 0$.

results for the linear approximation [first term on the right-hand side of Eq. (12), dot-dashed lines (virtually coinciding with the exact curve for all $B_2/E_F \lesssim 0.1$)], as well as for the quadratic approximation (dashed parabolas) as given by the leading term in Eq. (13) for stronger couplings. For sufficiently weak coupling, the exact dispersion relation is virtually linear—*in spite of the divergence* of the quadratic term in Eq. (12), as $B_2/E_F \to 0$. As E_F decreases, the quadratic dispersion relation (13) begins to dominate very slowly. A result unique to 2D (and associated with the fact that in 2D, the fermionic density of states is *independent* of energy) is that $-\mathcal{E}_0 \equiv \Delta_0 = B_2$. Similar results have recently been found[87] in 1D with the same delta-potential interaction assumed between electrons, except that the linear term is simply $\hbar v_F K$ for weak coupling, and for *any* coupling, the dispersion relation in the vacuum limit $v_F \to 0$ is again the expected quadratic $\hbar^2 K^2/2(2m)$.

In 3D, instead of Eq. (11), similar procedures[88] for two-spin fermions give

$$\sum_{\mathbf{k}} \frac{1}{\hbar^2 k^2/m} - \sum_{\mathbf{k},(|\mathbf{k}\pm\mathbf{K}/2|>k_F)} \frac{1}{\hbar^2 k^2/m - \mathcal{E}_K - 2E_F + \hbar^2 K^2/4m} = \frac{mL^3}{4\pi\hbar^2}\frac{1}{a} \quad (14)$$

where a is the s-wave scattering length associated with the regularized δ-well, which corresponds with weak to strong coupling accordingly as $-\infty < 1/k_F a < +\infty$. One

finds that for weak coupling ($k_F a \to 0^-$, e.g., prior to the well-known first-bound-state singularity as the depth of a 3D potential well is increased) that

$$\mathcal{E}_0/E_F \to -(8/e^2)\exp(-\pi/k_F|a|) \tag{15}$$

a result first reported by Van Hove.[89] For strong coupling ($k_F a \to 0^+$, i.e., just beyond the single-bound-state resonance) one gets

$$\mathcal{E}_0/E_F \to -2/(k_F a)^2\,. \tag{16}$$

Numerical results[88] in 3D, very similar to those in Fig. 5 for 2D are obtained. Namely, for weak coupling, the CP dispersion curves are very nearly linear while for smaller density, they very slowly tend to the quadratic. The limit given by Eq. (12) in 2D was found to be too complicated in 3D to be evaluated analytically, except for weak coupling. Repeating the 2D analysis without attempting to explicitly determine the coefficient of the quadratic term, one gets, for the δ-well interfermion interaction in weak coupling,

$$\varepsilon_K \equiv \mathcal{E}_K - \mathcal{E}_0 = \frac{1}{2}\hbar v_F K + O(K^2)\,. \tag{17}$$

This is the same result cited without proof by Schrieffer in 1964 (Ref. 3, p. 33) for the Cooper model interaction, to which we now turn.

3.2. *Cooper model interaction between electrons*

The Cooper model interaction mimics, in a simple way, the attractive electron-phonon interaction plus the repulsive Coulomb interaction and has the form of Eq. (7) with $V > 0$ and $\hbar\omega_D \equiv \hbar^2 k_D^2/2m$ is the maximum energy of a vibrating-ionic-lattice phonon. This means that two fermions interact with a constant attraction $-V$ when the tip of their relative-momentum wavevector $\mathbf{k}$ points anywhere inside the overlap volume in k-space of the two spherical shells in Fig. 2. Inserting Eq. (7) into Eq. (4) and converting sums over $\mathbf{k}$ into energy integrals by introducing the electronic density of states (DOS) $N(\epsilon)$ for each spin gives

$$\begin{aligned} 1 &= V \sum_{\mathbf{k}}{}' [2\epsilon_k - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m]^{-1} \\ &= V \int_{E_F}^{E_F + \hbar\omega_D} \frac{N(\epsilon)d\epsilon}{2\epsilon - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m}\,. \end{aligned} \tag{18}$$

From this, one immediately obtains the familiar result for $K = 0$ stated initially in Eq. (1). The equality in Eq. (1) is *exact* in 2D for all coupling — as well as in 1D or 3D provided that $\hbar\omega_D \ll E_F$ so that $N(\epsilon) \simeq N(E_F)$, a constant that can be taken outside the integral in Eq. (18).

For a 2D system, Eq. (18) gives[90] for weak coupling

$$\mathcal{E}_K \xrightarrow[K\to 0]{} \mathcal{E}_0 + (2/\pi)\hbar v_F K + O(K^2)\,. \tag{19}$$

The exact dispersion relation obtained numerically from Eq. (18) for $\lambda = 1/2$ and $\hbar\omega_D/E_F = 10^{-2}$ shows that the linear approximation (19) is very good for moderately small λ and $\hbar\omega_D/E_F$, over the entire range of K values for which $\mathcal{E}_K \leq 0$. Note that the linear term carries the *same* coefficient as Eq. (12) for a *different* interfermion interaction. Pair breakup, specifically $\mathcal{E}_K > 0$ for these values of λ and $\hbar\omega_D/E_F$, occurs at a relatively small value of K, about four orders of magnitude smaller than the maximum value $2\sqrt{k_F^2 + k_D^2}$ allowed by the interaction (7).

In 3D, assuming $\hbar\omega_D/E_F \ll 1$ so that the DOS $N(\epsilon) = \sqrt{1/2m^3}\epsilon/\pi^2\hbar^3$ can be replaced by $N(E_F)$ and then taken outside the integral sign, the result cited in Ref. 3, p. 33, (see also Ref. 75, p. 336, Prob. 10.4, but note here a coefficient of 1 instead of 1/2, perhaps misprinted) follows, namely

$$\mathcal{E}_K \underset{K\to 0}{\longrightarrow} \mathcal{E}_0 + \frac{1}{2}\hbar v_F K + O(K^2)\,. \tag{20}$$

Exact numerical results in 3D are qualitatively similar[90] to those in 2D with regards to the goodness of the linear approximation for weak coupling.

4. Cooper Pairing via Bethe-Salpeter Equation

The "original" CP problem just summarized for two distinct interfermion interactions (the δ-well and the Cooper model interaction), neglects the effect of two-hole (2h) CPs treated on an equal footing with two-particle (2p), or two-electron, CPs — as Green's functions[75] can naturally guarantee. The prime motivation rests on the recently established fact[36,37] that a BCS condensate is precisely a BE condensate with equal numbers of 2p and 2h CPs, in the limit of weak coupling. Further motivation comes from the unique but unexplained role that *hole* charge carriers seem to be playing in the normal state of superconductors[91] in general (see also Ref. 92). Final motivation stems from the ability of the "complete (in that both 2h- and 2p-CPs are allowed in varying proportions) BF model" of Refs. 36–38 to "unify" both BCS and BEC theories as special cases, and to predict substantially higher T_c's than BCS theory without abandoning electron-phonon dynamics. The latter is important as compelling evidence has recently been reported for a significant, if not sole, presence of phonons in high-T_c cuprate superconductors from angle-resolved photoemission spectroscopy (ARPES) data.[93]

In this section, we sketch how the Bethe-Salpeter (BS)[94] many-body integral equation (in the ladder approximation) treating both 2p and 2h pairs on an equal footing shows that, while the ordinary CP problem [based on an ideal Fermi gas (IFG) ground state (the usual "Fermi sea")] does *not* possess stable energy solutions, it does so when the IFG ground state is replaced by the BCS one. This is equivalent to starting from an unperturbed Hamiltonian that is the BCS ground state, instead of the pure-kinetic-energy operator corresponding to the IFG. We discuss how: (i) CPs not based on the IFG-sea but on the BCS ground state survive in a *nontrivial* solution as "generalized" or "moving" CPs which are *positive* energy resonances with an imaginary energy term leading to finite-lifetime effects; (ii) as

in the "ordinary" CP problem of the previous section, their dispersion relation in leading order in the total momentum (or CMM) $\hbar\mathbf{K} \equiv \hbar(\mathbf{k}_1 + \mathbf{k}_2)$ is also *linear*, rather than the quadratic $\hbar^2 K^2/2(2m)$ of a composite boson (e.g., a deuteron) of mass $2m$ moving not in the Fermi sea but in vacuum; and (iii) this latter "moving CP" solution, though often confused with it, is physically *distinct* from another more common *trivial* solution sometimes called the Anderson-Bogoliubov-Higgs (ABH),[95] (Ref. 51 p. 44),[96–98] collective excitation. Bogoliubov[49] seems to have been the first to derive this excitation spectrum. The ABH mode is also linear in leading order and goes over into the IFG ordinary sound mode in zero coupling. All this occurs in 2D[99] as well as in the 3D study outlined earlier in Ref. 100 and most recently also in 1D.[101] We focus here on 2D because of its interest[102–104] for quasi-2D high-T_c cuprate superconductors. In general, the results will be crucial for BEC scenarios employing BF models of superconductivity, not only *in exactly 2D* as with the Berezinskii-Kosterlitz-Thouless[105–107] transition, but also down to $(1+\epsilon)$D which characterize such quasi-1D superconductors such as the organometallics (Bechgaard salts)[108–110] and nanotubes.[111]

These results also apply, albeit with a different interaction, to neutral-fermion superfluidity, as in liquid ^{3}He,[112] and very probably also in ultracold trapped alkali Fermi gases such as ^{40}K[113] and ^{6}Li[114] atoms but where the role of hole-pairs is yet to be explored.

4.1. *Significance of Cooper instability*

In dealing with the many-electron system, we again assume, not the Cooper but the BCS model interaction (5) in 2D with double Fourier transform

$$\begin{aligned} \nu(k_1, k_1') &= -(k_F^2/k_1 k_1')V \quad && \text{if } k_F - k_D < k_1, k_1' < k_F + k_D \\ &= 0 && \text{otherwise.} \end{aligned} \tag{21}$$

Here $V > 0$, $\hbar k_F \equiv m v_F$ the Fermi momentum, m the effective electron mass, v_F the Fermi velocity, and $k_D \equiv \omega_D/v_F$ with ω_D the Debye frequency; note the difference with the previous definition just below Eq. (8). The usual physical constraint $\hbar\omega_D \ll E_F$ then implies that $k_D/k_F \equiv \hbar\omega_D/2E_F \ll 1$. Assuming perfect ph symmetry about the Fermi surface (or, alternatively, a very "flat" behavior of $\epsilon_k = \hbar^2 k^2/2m$ around the value $\epsilon_{k_F} \equiv E_F = \hbar^2 k_F^2/2m$) we set

$$\epsilon_k \simeq E_F + \hbar v_F (k - k_F) \tag{22}$$

as it simplifies all calculations when very near the Fermi surface.

The bound-state BS wavefunction equation[100] in the ladder approximation with both particles and holes for the original IFG-based CP problem is

$$\Psi(\mathbf{k},E) = -\left(\frac{i}{\hbar}\right)^2 G_0(\mathbf{K}/2 + \mathbf{k}, \mathcal{E}_K/2 + E) G_0(\mathbf{K}/2 - \mathbf{k}, \mathcal{E}_K/2 - E)$$

$$\times \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE' \frac{1}{L^d}\sum_{\mathbf{k}'} v(|\mathbf{k}-\mathbf{k}'|)\Psi(\mathbf{k}',E')\,. \tag{23}$$

Here L^d is the "volume" of the d-dimensional system; $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$ is the CMM wavenumber and $\mathbf{k} \equiv 1/2(\mathbf{k}_1 - \mathbf{k}_2)$ the relative wavevectors of the 2e bound state whose wavefunction is $\Psi(\mathbf{k}, E)$; $\mathcal{E}_K \equiv E_1 + E_2$ is the energy of this bound state while $E \equiv E_1 - E_2$, and $G_0(\mathbf{K}/2 + \mathbf{k}, \mathcal{E}/2 + E)$ is the bare one-fermion Green's function given by (Ref. 75, p. 72)

$$G_0(\mathbf{k}_1, E_1) = \frac{\hbar}{i}\left\{\frac{\theta(k_1 - k_F)}{-E_1 + \epsilon_{\mathbf{k}_1} - E_F - i\varepsilon} + \frac{\theta(k_F - k_1)}{-E_1 + \epsilon_{\mathbf{k}_1} - E_F + i\varepsilon}\right\} \tag{24}$$

where $\epsilon_{\mathbf{k}_1} \equiv \hbar^2 k_1^2/2m$ and $\theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$, so that the first term refers to *electrons* and the second to *holes*. The latter are also fermions but of positive charge 2e. Figure 6 shows all Feynman diagrams for the 2p, 2h and ph wavefunctions Ψ_+, Ψ_- and Ψ_0, respectively, that emerge in the general (BCS-ground-state-based) problem to be discussed later.

Let us first consider the case where all diagrams with holes in Fig. 6 are ignored, i.e., we neglect the second term in Eq. (24). Furthermore, we note that the energy dependence in Eq. (23) derives from the Green's function only, and therefore we may define a new function $\varphi(\mathbf{k})$ by first writing

$$\Psi(\mathbf{k}, E) \equiv G_0(\mathbf{K}/2 + \mathbf{k}, \mathcal{E}_K/2 + E)G_0(\mathbf{K}/2 - \mathbf{k}, \mathcal{E}_K/2 - E)\varphi(\mathbf{k}) \tag{25}$$

which upon substitution in Eq. (23) yields

$$\begin{aligned}\varphi(\mathbf{k}) = -\frac{1}{L^d}\sum_{\mathbf{k}'} v(|\mathbf{k}-\mathbf{k}'|)\varphi(\mathbf{k}')\left(\frac{i}{\hbar}\right)^2 \frac{1}{2\pi i}\int_{-\infty}^{+\infty} dE' \\ \times \frac{\theta(k'-k_F)}{-\mathcal{E}_K/2 - E' + \epsilon_{\mathbf{K}/2+\mathbf{k}'} - E_F - i\varepsilon} \\ \times \frac{\theta(k'-k_F)}{-\mathcal{E}_K/2 + E' + \epsilon_{\mathbf{K}/2-\mathbf{k}'} - E_F - i\varepsilon}\,.\end{aligned} \tag{26}$$

The energy integration then leaves

$$\varphi(\mathbf{k}) = -\frac{1}{L^d}\sum_{\mathbf{k}'} v(|\mathbf{k}-\mathbf{k}'|)\frac{\theta(k'-k_F)}{\epsilon_{\mathbf{K}/2+\mathbf{k}'} + \epsilon_{\mathbf{K}/2-\mathbf{k}'} + 2E_F - \mathcal{E}_K}\varphi(\mathbf{k}') \tag{27}$$

which may be recognized as the Bethe-Goldstone equation.[115] It can be shown to follow from taking just the first two diagrams on the rhs of the top line in Fig. 6.

For the IFG-based scenario for CPs when holes are *not* neglected, diagrams enclosed in rectangles do not contribute, as they involve factors of $\theta(k_1 - k_F)\theta(k_F - k_1) \equiv 0$. Since the energy dependence of $\Psi(\mathbf{k}, E)$ in Eq. (23) is only through the Green's functions, the ensuing energy integrals (23) can be evaluated directly in the complex E'-plane and yield, for interaction (21), an equation for the wavefunction

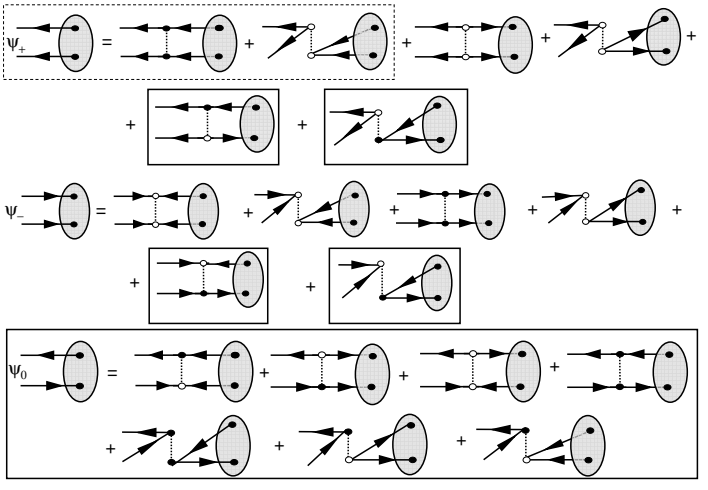

Fig. 6. Wavefunction Feynman diagrams for $2e$ (Ψ_+), $2h$ (Ψ_-) and eh (Ψ_0) pair states arising from the BS[94] equations. The first three diagrams (enclosed in the dashed-lined rectangle) associated with the 2e wavefunction Ψ_+ correspond to the direct and exchange diagrams of the Bethe-Goldstone equation[115] that describes the original Cooper problem. Full-lined rectangles contain diagrams that do *not* contribute in the IFG-based case, as explained in the text.

$\psi_{\mathbf{k}}$ in momentum space for CPs with *zero* CMM $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2 = 0$ that is, if $\xi_k \equiv \hbar^2 k^2/2m - E_F$,

$$(2\xi_k - \mathcal{E}_0)\psi_{\mathbf{k}} = V \sum_{\mathbf{k}'}{}' \psi_{\mathbf{k}'} - V \sum_{\mathbf{k}'}{}'' \psi_{\mathbf{k}'} \tag{28}$$

where $\mathcal{E}_0$ is the $\mathbf{K} = 0$ eigenvalue energy, and $1/2(\mathbf{k}_1 - \mathbf{k}_2) = \mathbf{k}_1$. The single prime over the first (2p-CP) summation term denotes the restriction $0 < \xi_{k'} < \hbar\omega_D$, while the double prime in the last ($2h$-CP) term means $-\hbar\omega_D < \xi_{k'} < 0$. Without this latter term, we have Cooper's Schrödinger-like equation[2] in momentum representation for 2p-CPs whose implicit solution is clearly $\psi_{\mathbf{k}} = (2\xi_k - \mathcal{E}_0)^{-1} V \sum_{\mathbf{k}'}{}' \psi_{\mathbf{k}'}$. Since the summation term is constant, performing that summation on both sides allows the cancellation of the $\psi_{\mathbf{k}}$-dependent terms, leaving the eigenvalue equation $\sum_{\mathbf{k}}{}' (2\xi_k - \mathcal{E}_0)^{-1} = 1/V$. This is one equation in one unknown $\mathcal{E}_0$; transforming the sum to an integral over energies gives the familiar solution (1), where as before $\lambda \equiv VN(E_F)$ with $N(E_F)$ being the electronic DOS for one spin. This solution is exact in 2D, and to a very good approximation otherwise if $\hbar\omega_D \ll E_F$. It corresponds to a negative-energy, stationary-state bound pair. For $K \geqslant 0$, the 2p-CP eigenvalue equation becomes

$$\sum_{\mathbf{k}}{}' (2\xi_k - \mathcal{E}_K + \hbar^2 K^2/4m)^{-1} = 1/V\,. \tag{29}$$

Note that a 2p-CP state of energy $\mathcal{E}_K$ is characterized only by a definite K but *not* definite $\mathbf{k}$, in contrast to a "BCS pair" defined [Ref. 1, Eqs. (2.11) to (2.13)] with fixed $\mathbf{K}$ and $\mathbf{k}$ (or equivalently definite $\mathbf{k}_1$ and $\mathbf{k}_2$). For more details see Ref. 73.

Without the first summation term in Eq. (28), the same expression (1) for the $\mathcal{E}_0$ of 2p-CPs follows for $2h$-CPs, apart from an overall sign change.

In spite of its simple appearance, the *complete* Eq. (28) *cannot* be derived from an ordinary (non-BS) Schrödinger-like equation (in the momentum representation). To solve Eq. (28) for the unknown energy $\mathcal{E}_0$, let the rhs of Eq. (28) be defined as $A-B$, with A relating to the 2p term and B to the $2h$ one. Solving for the unknown $\psi_{\mathbf{k}}$ gives

$$\psi_{\mathbf{k}} = (A-B)/(2\xi_k - \mathcal{E}_0) \;\text{ or equivalently }\; \psi(\xi) = (A-B)/(2\xi - \mathcal{E}_0) \tag{30}$$

whence

$$A \equiv \lambda \int_0^{\hbar\omega_D} d\xi \psi(\xi) = \frac{1}{2}(A-B)\lambda \int_{-\mathcal{E}_0}^{2\hbar\omega_D - \mathcal{E}_0} dz/z \equiv (A-B)x \tag{31}$$

$$B \equiv \lambda \int_{-\hbar\omega_D}^{0} d\xi \psi(\xi) = \frac{1}{2}(A-B)\lambda \int_{-2\hbar\omega_D - \mathcal{E}_0}^{-\mathcal{E}_0} dz/z \equiv (A-B)y\,. \tag{32}$$

The integrals are readily evaluated giving $x \equiv 1/2\lambda \ln(1 - 2\hbar\omega_D/\mathcal{E}_0)$ and $y \equiv -1/2\lambda \ln(1+2\hbar\omega_D/\mathcal{E}_0)$. As A and B still contain the unknown $\psi(\xi)$, let us eliminate them. Note that Eq. (31) and Eq. (32) are equivalent to *two* equations in two unknowns A and B, namely

$$(1-x)A + xB = 0$$

$$-yA + (1+y)B = 0\,.$$

This immediately leads to the equation $1 - x + y = 0$, which on inserting the definitions of x and y becomes

$$1 = \frac{1}{2}\lambda \ln[1 - (2\hbar\omega_D/\mathcal{E}_0)^2]$$

which finally yields Eq. (2) on solving for $\mathcal{E}_0$. As the CP energy $\mathcal{E}_0$ under these conditions is *pure-imaginary*, there is obviously a serious instability of the CP problem when both particle- and hole-pairs are included and at the same time referred to the IFG ground-state. This was reported in Ref. 51 p. 44 and Ref. 64, which did not, however, stress the pure $2p$ and $2h$ special cases just discussed. It was also derived heuristically (Ref. 3 p. 168) without formally referring to $2h$-CPs as objects independent of $2e$-CPs. Clearly then, the original CP picture *is meaningless if particle- and hole-pairs are treated on an equal footing* as completeness and consistency demands. The result (2) is a *true* instability, as it entails an energy without a real part whatsoever, in contrast to the *benign* instability (1) which merely signals a negative-energy bound state. Thus, Eq. (2) means that one either scraps the concept of a CP or replaces the unperturbed IFG zeroeth-order state by a different one in an attempt to formulate CPs properly.

5. Cooper Pairing in BCS-Correlated "Sea"

However, a BS treatment not about the IFG sea but about the BCS ground state *vindicates the CP concept* as a nontrivial solution. This is equivalent to starting not from the IFG unperturbed Hamiltonian but from the BCS one. Its physical justification lies in recovering three expected results: the ABH sound mode, the BCS $T = 0$ gap equation and finite-lifetime effects of the "moving CPs." In either 2D[99] or 3D[100] or 1D,[101] the BS equation yields a 4×4 determinant which can be shown to reduce to two blocks, a 3×3 and a 1×1 determinant. The two determinants represent *two distinct solutions*: a) the trivial ABH sound solution and b) a highly *nontrivial* moving CP solution, respectively. In either case, the BS formalism initially gives a set of three coupled equations, one for each ($2p$, $2h$ and ph) channel wavefunction for any spin-independent interaction such as Eq. (21). However, the ph channel carries no current; moreover, it decouples from the $2p$ and $2h$ equations, leaving only two coupled wavefunction equations for the ABH solution, which we examine first. In the study in Ref. 116, the hh channel was explicitly ignored, leading to a 3×3 determinant from which only the trivial ABH "sound" solution emerges, so that the nontrivial moving CP "particle" solution appears to have been overlooked as a separate solution entirely. In contrast with the present case of a Fermi gas, in the pure boson gas, both "particle" and "sound" solutions are indistinguishable.[71,72] In the fermion case, it should be feasible to search for, identify and distinguish both particle and sound modes experimentally, e.g., see Refs. 117 and 118.

The IFG Green function (24) is now replaced by the BCS one

$$\mathbf{G}_0(\mathbf{k}_1, E_1) = \frac{\hbar}{i}\left\{\frac{v_{k_1}^2}{-E_1 + E_{k_1} - i\varepsilon} + \frac{u_{k_1}^2}{-E_1 + E_{k_1} + i\varepsilon}\right\} \tag{33}$$

where $E_k \equiv \sqrt{\xi_k^2 + \Delta^2}$ with Δ the $T = 0$ fermionic gap, $v_k^2 \equiv 1/2(1 - \xi_k/E_k)$ and $u_k^2 \equiv 1 - v_k^2$ are the BCS-Bogoliubov-Valatin[119,120] function-coefficients. They are (Ref. 121, p. 135), respectively, the probability that an electron is occupied or unoccupied. As $\Delta \to 0$, these three quantities become $|\xi_k|$, $\theta(k_1 - k_F)$ and $\theta(k_F - k_1)$, respectively, so that Eq. (33) reduces to Eq. (24), as expected. Substituting $G_0(\mathbf{k}_1, E_1)$ by $\mathbf{G}_0(\mathbf{k}_1, E_1)$ corresponds to rewriting the total Hamiltonian so that the pure-kinetic-energy unperturbed Hamiltonian is replaced by the BCS one. The remaining Hamiltonian terms are then assumed to be amenable to a perturbation treatment. We focus again in this section only on 2D.

5.1. *ABH sound (trivial) solution*

The equations involved are too lengthy even in 2D, and will be derived in detail elsewhere, but for the trivial ABH sound solution, the aforementioned 3×3 determinant boils down[99] to the single expression

$$\frac{1}{2\pi}\lambda\hbar v_F \int_{k_F - k_D}^{k_F + k_D} dk \int_0^{2\pi} d\varphi \{u_{\mathbf{K}/2+\mathbf{k}} u_{\mathbf{K}/2-\mathbf{k}} + v_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}}\}$$

$$\times \left[\frac{v_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}}}{\mathcal{E}_K + E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}}} + \frac{u_{\mathbf{K}/2+\mathbf{k}}\, u_{\mathbf{K}/2-\mathbf{k}}}{-\mathcal{E}_K + E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}}} \right] = 1 \qquad (34)$$

where φ is the angle between $\mathbf{K}$ and $\mathbf{k}$. Here $k_D \equiv \omega_D/v_F$; note the difference with the definition just below Eq. (8). As before, $\lambda \equiv VN(E_F)$ with $N(E_F) \equiv m/2\pi\hbar^2$ being the constant 2D electronic DOS and $V > 0$ is the interaction strength defined in Eq. (21). It is worth mentioning that ARPES studies of BiSrCaCuO have shown direct evidence[122] in this cuprate for the BCS-Bogoliubov-Valatin functions u_k^2 and v_k^2, both above and below the Fermi energy.

The ABH collective excitation mode energy $\mathcal{E}_K$ must then be extracted from this equation. For $\mathbf{K} = 0$, it is just $\mathcal{E}_0 = 0$, (Ref. 51 p. 39). Then Eq. (34) rewritten as an integral over $\xi \equiv \hbar^2 k^2/2m - E_F$ reduces to

$$\int_0^{\hbar\omega_D} d\xi/\sqrt{\xi^2 + \Delta^2} = 1/\lambda \qquad (35)$$

or the familiar BCS $T = 0$ gap equation for interaction (21). The integral is exact and gives

$$\Delta = \hbar\omega_D/\sinh(1/\lambda)\,. \qquad (36)$$

Returning to the ABH energy $\mathcal{E}_K$ equation (34) and Taylor-expanding $\mathcal{E}_K$ about $K = 0$ and small λ leaves

$$\mathcal{E}_K = \frac{\hbar v_F}{\sqrt{2}} K + O(K^2) + o(\lambda) \qquad (37)$$

where $o(\lambda)$ denote interfermion interaction correction terms that vanish as $\lambda \to 0$. Note that the leading term is just the ordinary sound mode in an IFG whose sound speed in d dimensions is simply $c = v_F/\sqrt{d}$. In 3D, the term $\sqrt{2}$ in Eq. (37) was found[100] to be replaced by $\sqrt{3}$.

This result also follows elementarily on solving for c in the familiar thermodynamic relation $dP/dn = mc^2$ involving the zero-temperature IFG pressure

$$\begin{aligned} P &= n^2[d(E/N)/dn] = 2nE_F/(d+2) \\ &= 2C_d n^{2/d+1}/(d+2) \end{aligned} \qquad (38)$$

where the d-dependent constant C_d will drop out. Here, the IFG ground-state (internal) energy per fermion $E/N = dE_F/(d+2) = C_d n^{2/d}$ was used along with the Fermi energy $E_F \equiv \hbar^2 k_F^2/2m$, as well as

$$n \equiv N/L^d = k_F^d/d2^{d-2}\pi^{d/2}\Gamma(d/2) \qquad (39)$$

for the fermion-number density n. The derivative of Eq. (38) with respect to n finally gives $c = \hbar k_F/m\sqrt{d} \equiv v_F/\sqrt{d}$ in any d, and which in 2D is just the leading term in Eq. (37).

5.2. *Moving-CP (nontrivial) solution*

The nontrivial *moving-CP* solution of the BCS-ground-state-based BS treatment, which is relatively *new* (see also Refs. 97 and 98), comes from the remaining 1×1 determinant. It leads to the pair energy $\mathcal{E}_K$, which in 2D is contained in the equation[99]

$$\frac{1}{2\pi}\lambda\hbar v_F \int_{k_F-k_D}^{k_F+k_D} dk \int_0^{2\pi} d\varphi u_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}} \times \{u_{\mathbf{K}/2-\mathbf{k}} v_{\mathbf{K}/2+\mathbf{k}} - u_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}}\} \frac{E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}}}{-\mathcal{E}_K^2 + (E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}})^2} = 1\,. \quad (40)$$

In addition to the pp and hh wavefunctions (depicted diagrammatically in Ref. 100 Fig. 2 for the 3D case), diagrams associated with the ph channel give zero contribution at $T = 0$. A third equation for the ph wavefunction describes the ph bound state but turns out to depend only on the pp and hh wavefunctions. As the ph mode does not carry electric current, it will be ignored here. Taylor-expanding $\mathcal{E}_K$ in Eq. (40) in powers of K around $K = 0$, and introducing a possible damping factor by adding an imaginary term $-i\Gamma_K$ in the denominator, yields to order K^2

$$\pm\mathcal{E}_K \simeq 2\Delta + \frac{\lambda}{2\pi}\hbar v_F K + \frac{1}{9}\frac{\hbar v_F}{k_D} e^{1/\lambda} K^2 - i\left[\frac{\lambda}{\pi}\hbar v_F K + \frac{1}{12}\frac{\hbar v_F}{k_D} e^{1/\lambda} K^2\right] + O(K^3) \quad (41)$$

where the upper and lower signs refer to $2p$- and $2h$-CPs, respectively. A linear dispersion in leading order appears again, but is now associated with the bosonic moving CP. Hence the *positive*-energy 2p-CP resonance has a width Γ_K that vanishes as $K \to 0$. It has a lifetime

$$\tau_K \equiv \hbar/2\Gamma_K = \hbar/2[(\lambda/\pi)\hbar v_F K + (\hbar v_F/12k_D)e^{1/\lambda}K^2] \quad (42)$$

that diverges only at $K = 0$, falling to zero as K increases. Thus, "faster" moving CPs are shorter-lived and eventually break up, while "non-moving" ones are infinite-lifetime stationary states. The linear term $(\lambda/2\pi)\hbar v_F K$ in Eq. (41) in 2D, or $(\lambda/4)\hbar v_F K$ in 3D,[100] contrasts sharply with the *coupling-independent* leading-term $(2/\pi)\hbar v_F K$ from Eq. (19) in 2D, or $(1/2)\hbar v_F K$ from Eq. (20) in 3D, that follow from the *original* CP problem (29) neglecting holes. Note that the latter coupling-independent linear terms hold for either interaction (21)[90] *or* for an attractive delta interfermion potential well in 2D[85] or 3D,[88] respectively. The 2D and 3D δ-potential-wells are imagined regularized[84] to possess a single bound level whose binding energy (in 2D) or scattering length (in 3D) serve as the coupling parameter.

Figure 7(a) shows the exact "moving-CP" (mCP) energy (full curves) extracted from Eq. (40), along with its leading linear-dispersion term (thin short-dashed lines)

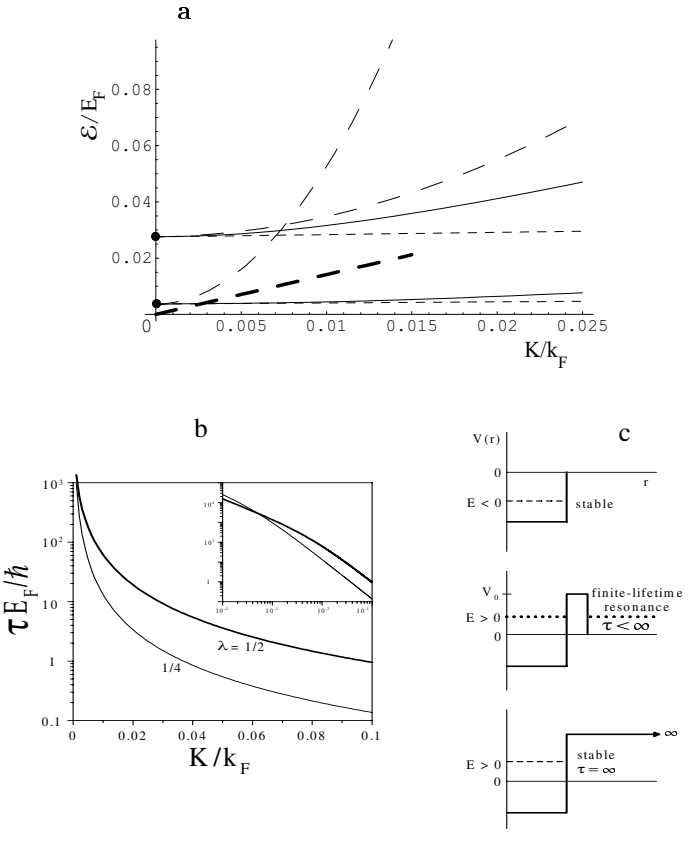

Fig. 7. Exact 2p-(or 2e-)moving-CP (real) energy $\mathcal{E}_K$ (in units of E_F) in 2D from Eq. (40) (full curves), compared with its linear leading term (thin short-dashed lines) and its linear plus quadratic expansion (long-dashed curves) both from (41), versus CMM wavenumber K (in units of k_F), for interaction (21) parameters $\lambda = 1/4$ (lower set of curves) and $1/2$ (upper set of curves), and $\hbar\omega_D/E_F = 0.05$. For reference, leading linear term (37) of so-called "trivial" ABH sound mode is also plotted (lower thick dashed line). (b) 2p-moving-CP lifetime as defined in Eq. (42). (c) Analogy of ordinary and BCS-based-BS 2p-moving-CPs with various confined states in a 3D potential-well problem.

and this plus the next (quadratic) term (long-dashed curves) from Eq. (41). The interaction parameter values used with Eq. (21) were $\hbar\omega_D/E_F = 0.05$ (a typically lower value for cuprates) and the two values $\lambda = 1/4$ and $1/2$. Using Eq. (36) in Eq. (41) gives

$$\mathcal{E}_0/E_F \equiv 2\Delta/E_F = 2\hbar\omega_D/E_F \sinh(1/\lambda) \tag{43}$$

having the values $\simeq 0.004$ and 0.028, respectively (marked as dots on the figure ordinate). Remarkably enough, the linear approximation (thin short-dashed lines in figure) is better over a wider range of K/k_F values for weaker coupling (lower set of three curves) in spite of a larger and larger (because of the factor $e^{1/\lambda}$) partial contribution from the quadratic term in Eq. (41). This peculiarity also emerged

from the ordinary CP treatment of Sec. 3, Refs. 85, 88 and 90. It suggests the expansion in powers of K to be an asymptotic series that should be truncated after the linear term. For reference, we also plot the linear leading term $\hbar v_F K/\sqrt{2}$ of the sound solution (37). We note that the *coupling-independent* leading-term[90] $(2/\pi)\hbar v_F K$ from the *original* CP problem neglecting holes, if graphed in Fig. 7(a), would almost coincide with the ABH term $\hbar v_F K/\sqrt{2}$ but have a slope about 90% smaller.

These results complete our understanding of CPs referred to a BCS-correlated ground-state in 1D,[101] 2D[99] and 3D[100] and are summarized in Table 1 below for the BCS model interaction in weak coupling, where we also include previous results for $\mathcal{E}_K$ in lowest order in K and weak coupling. Note that the first column (BG equation) refers to results ignoring $2h$-CPs and predict negative-energy *bound* pair states, while the second column (BS equation) refers to results where both kinds of pairs are included and correspond to positive-energy *resonant* pair states associated with $2e$-CPs. It is worth noting that the 3D entry in the first column was already reported (without proof) in Ref. 3 p. 33. For $2h$-moving-CPs, all entries in the second column carry a sign change. A crucial difference between the first and second columns is the coupling-(λ)-independent linear dispersion of the ordinary CP problem, as envisaged in the usual BG formulation, first column. A critical observation is that a binary boson-fermion gas formulation of BEC critical T_c's shows that T_c *vanishes* smoothly[123] as $\lambda \to 0$ (as it should since there will be *no bosons left* in the BF mixture) *only* for the more general formulation based on the BS equation when both kinds of pairs are present. For the BG result (ordinary Cooper pair), the BEC T_c approaches an unphysical *finite* value as $\lambda \to 0$, perhaps an artifact of the CP problem if hole-pairs are ignored.

The interactionless ABH mode should reduce to the ideal Fermi gas (IFG) in any dimensionality d. The results just below Eq. (39) are consistent with the results for $d = 1, 2$ and 3 found from the BS approach and tabulated in the last column of Table 1.

As in Cooper's[2] original equation (29), the BS moving CPs for either $2e$ or $2h$ pairs are characterized by a definite $\mathbf{K}$ and *not* also by definite $\mathbf{k}$ as the pairs discussed by BCS.[1] Hence, the objection does not apply that CPs are not bosons because BCS pairs with definite $\mathbf{K}$ and $\mathbf{k}$ (or equivalently definite $\mathbf{k}_1$ and $\mathbf{k}_2$) have creation/annihilation operators that do *not* obey the usual Bose commutation relations [Ref. 1, Eqs. (2.11) to (2.13)]. In fact, either eigenvalue equation for $\mathcal{E}_K$ (29) or (40) shows that a given "ordinary" (29) or "generalized" CP (40) state labeled by either $\mathbf{K}$ or $\mathcal{E}_K$ can accommodate (in the thermodynamic limit) an indefinitely large number of possible BCS pairs with different $\mathbf{k}$'s.[73] A recent electronic analog[125] of the Hanbury Brown–Twiss photon-effect experiment suggests electron pairs in a SC to definitely be bosons.

To summarize this section, hole ($2h$) pairs treated on par with electron ($2e$) pairs play a vital role in determining the precise nature of CPs even at zero temperature — only when based not on the usual IFG "sea" but on the BCS ground state. Their

Table 1. Small CMM K, weakly-coupled, *moving*-2e-CP energies for the Cooper model interaction (BG = Bethe-Goldstone case) and for the BCS model interaction (BS = Bethe-Salpeter case) while the ABH mode refers to a sound excitation where only the coupling-λ-independent leading term is given and coincides with the ideal Fermi gas result. In the moving-CP BS case, Δ is the $T = 0$ BCS energy gap and the moving-2h-CP solution is the same as for the moving-2e-CPs but with all signs changed.

d	BG (2e-CPs only)	BS (2e-moving-CPs)	BS (ABH mode)
1D	$-2\hbar\omega_D e^{-2/\lambda} + \hbar v_F K + \cdots$ Ref. 87 and 124	$2\Delta + \frac{1}{2}\lambda\hbar v_F K + \cdots$ Ref. 101	$\hbar v_F K + \cdots$ Ref. 101
2D	$-2\hbar\omega_D e^{-2/\lambda} + \frac{2}{\pi}\hbar v_F K + \cdots$ Ref. 90	$2\Delta + \frac{1}{2\pi}\lambda\hbar v_F K + \cdots$ Ref. 99	$\frac{1}{\sqrt{2}}\hbar v_F K + \cdots$ Ref. 99
3D	$-2\hbar\omega_D e^{-2/\lambda} + \frac{1}{2}\hbar v_F K + \cdots$ Ref. 90	$2\Delta + \frac{1}{4}\lambda\hbar v_F K + \cdots$ Ref. 100	$\frac{1}{\sqrt{3}}\hbar v_F K + \cdots$ Ref. 100

treatment with the Bethe–Salpeter integral equation in the ladder approximation gives purely-imaginary-energy CPs when referred to the IFG, and positive-energy resonant-state CPs with a finite lifetime for nonzero CMM when referred to the BCS ground state — instead of the more familiar negative-energy stationary states of the original IFG-based CP problem that neglects hole pairs, as sketched in arriving at Eq. (28) *without* the last term. The BS "moving-CP" dispersion relation (41), on the other hand, resembles the plasmon dispersion curve in 3D. It is gapped by twice the BCS energy gap, followed by a *linear* leading term in the CMM expansion about $K = 0$, instead of the well-known quadratic rise for the 3D plasmon dispersion curve. This linearity is distinct from the better-known one (37) associated with the sound or ABH collective excitation mode whose energy vanishes at $K = 0$. Thus, BF models assuming this CP linearity for the boson component, instead of the quadratic $\hbar^2 K^2/2(2m)$ assumed in Refs. 23–33, 36, 37, 42, 126–129 among many others, can give BEC for all $d > 1$, including exactly 2D where the BKT transition applies in principle. Such BF models can then, also in principle, address not only quasi-2D cuprate but also quasi-1D organometallic and nanotube superconductors.

6. Evidence for Linearly-Dispersive CPs

A BEC model was applied by Rosencwaig[130] using the quadratic CP dispersion relation to address seven cuprate superconductors (SCs) with transition temperatures T_c at optimal doping ranging from 22 K to 133 K. These are: $La_{2-x}Sr_xCuO_4$ (LSCO), $Nd_{2-x}Ce_xCuO_4$ (NCCO), $YBa_2Cu_3O_{7-y}$ (Y123), $Bi_2Sr_2CaCu_2O_{8-y}$ (Bi2212), $Bi_2Sr_2Ca_2Cu_3O_{10-y}$ (Bi2223), $HgBa_2CaCu_2O_{7-y}$ (Hg1212) and $HgBa_2Ca_2Cu_3O_{9-y}$ (Hg1223). On the other hand, empirical evidence for the *linearly-dispersive* nature of CPs in cuprates has been argued by Wilson[131] to be suggested by the scanning tunneling microscope conductance scattering data in BSCCO obtained by Davis and coworkers.[132–134] More suggestive direct evidence for this linearity is shown in Fig. 1 of Ref. 135 with experimental data (mostly from penetration-depth measurements) for two 3D SCs,[136,137] two quasi-2D cuprates,[138–140] and a quasi-1D nanotube SC.[141] The data are seen to agree quite well for $d = 3$, 2 and 1, at least for $T \gtrsim 0.5T_c$, with the *pure-phase* (only $2e$- or $2h$-CP) BEC condensate fraction formula associated[52] with the general dimensionality d- and dispersion-relation-exponent s-dependent BEC T_c-formula of Ref. 52, namely

$$1 - (T/T_c)^{d/s} \tag{44}$$

provided one assumes $s = 1$, in contrast to the $s = 2$ case studied in Ref. 130. For the cuprate data, there is a peculiar "changeover" from the $d/s = 3/2$ behavior predicted in Ref. 130 below $T/T_c \sim 1/2$ to $d/s = 2/1$ behavior above $T/T_c \sim 1/2$. For lower T's, one might argue[37] that a *mixed* BEC phase containing both $2e$- and $2h$-CPs becomes more stable (i.e., has lower Helmholtz free energy), as well as the influence of d-wave CP symmetry besides the s-wave one assumed throughout this paper, so that the simple pure-phase formula (44) is no longer strictly valid.

Indeed, formulae such as (19), (20) or (41) that rely on the explicit presence of a Fermi sea apply to the CPs in the binary boson-fermion gas mixture because only a minuscule fraction ($< 0.1\%$[142]) of the individual fermion charge carriers are paired up into CPs. This occurs for a Cooper model interaction forming the bosonic CPs with a maximum allowed[143] (see also Ref. 42 p. 204) coupling of $\lambda = 1/2$; see, however, reservations expressed in Ref. 144 on electron-phonon coupling upper limits. This ensures that a substantial Fermi sea is still present in the mixture. Such tiny fractions of paired charges are consistent with some very recent far-infrared charge-dynamics measurements[145,146] in LSCO. For more details, see Ref. 147.

7. Conclusions

Ordinary Cooper pairing of two electrons based on a Schrödinger-like equation in momentum space, and then generalized Cooper pairing including two-hole pairs via the Bethe-Salpeter (BS) integral equation in the ladder approximation, were first surveyed. The former deals with electron pairs only. These are describable in terms of the Bethe-Goldstone equation which is a special case of the BS equation. The BS equation handles hole pairs on an equal footing with electron pairs. When referred to the pure-kinetic-energy unperturbed Hamiltonian, i.e., the ideal Fermi gas, pairing of just one kind gives the familiar real-valued, negative, bound-state energy of a CP. But pairing with both kinds becomes meaningless as it leads to purely-imaginary energies. These constitute a *true* Cooper instability — in contrast to the "benign" instability that negative-energy bound states imply in the $2e$-CP problem. The true instability can be avoided entirely if the pairs are referred to the BCS-correlated ground state as unperturbed Hamiltonian instead of to the ideal Fermi gas sea as in the original Cooper formulation. As a result, besides a trivial solution sound-mode, both electron- and hole-pairing give rise in 1D, 2D and 3D to physically meaningful, i.e., *real-energy* electron-pair resonances with a small imaginary part implying a lifetime for CMM wavenumber $K > 0$ that is finite, and infinite for $K = 0$.

Acknowledgments

The authors thank B. L. Györffy and J. A. Wilson for enlightening discussions, as well as the Royal Society (UK) and the Mexican Academy of Sciences (Mexico) for a travel grant. MdeLl thanks the Texas Center for Superconductivity of the University of Houston (USA) for its hospitality when part of this work was carried out. He also thanks V. C. Aguilera-Navarro, P. W. Anderson, J. Batle, M. Casas, C. W. Chu, J. R. Clem, J. D. Dow, M. Fortes, M. Grether, R. Hulet, K. Levin, T. A. Mamedov, F. J. Sevilla, M. A. Solís, S. Tapia, C. S. Ting, V. V. Tolmachev, O. Rojo, J. J. Valencia, A. A. Valladares and H. Vucetich for invaluable discussions and/or for providing material prior to its publication, and acknowledges UNAM-DGAPA-PAPIIT (Mexico) grant #IN106401 as well as CONACyT (Mexico) grant #41302F for partial support.

References

1. J. Bardeen, L. N. Cooper and J. R. Schrieffer, *Phys. Rev.* **081**, 1175 (1957).
2. L. N. Cooper, *Phys. Rev.* **104**, 1189 (1956).
3. J. R. Schrieffer, *Theory of Superconductivity* (Benjamin, Reading, MA, 1983).
4. J. G. Bednorz and K. A. Müller, *Z. Phys.* **64**, 189 (1986).
5. M. K. Wu, J. R. Ashburn, C. J. Torng, P. H. Hor, R. L. Meng, L. Gao, Z. J. Huang, Y. Q. Wang and C. W. Chu, *Phys. Rev. Lett.* **58**, 908 (1987).
6. L. Gao, Y. Y. Xue, F. Chen, Q. Xiong, R. L. Meng, D. Ramírez, C. W. Chu, J. H. Eggert and H. K. Mao, *Phys. Rev. B* **50**, 4260 (1994).
7. C. Buzea and K. Robbie, *Supercond. Sci. Technol.* **18**, R1 (2005).
8. M. K. Wu, D. Y. Chen, F. Z. Chien, S. R. Sheen, D. C. Ling, C. Y. Tai, G. Y. Tseng, D. H. Chen and F. C. Zhang, *Z. f. Physik* **102**, 37 (1996).
9. D. Y. Chen, F. Z. Chien, D. C. Ling, J. L. Tseng, S. R. Sheen, M. J. Wang and M. K. Wu, *Physica C* **282–287**, 73 (1997).
10. H. A. Blackstead, *et al.*, *Cubic* $Ba_2YRu_{1-x}Cu_xO_6$*: A Possible Strong-Coupled SDW Superconductor* (unpublished).
11. P. W. Anderson, *The Theory of Superconductivity in the High* T_c *Cuprates* (Princeton University Press, Princeton, NJ, 1997).
12. P. A. Lee, N. Nagaosa and X. G. Wen, *Rev. Mod. Phys.* **78**, 17 (2006).
13. N. Bulut, *Adv. Phys.* **51**, 1587 (2002).
14. D. Manske, *Theory of Unconventional Superconductors: Cooper Pairing Mediated by Spin Excitations* (Springer Tracts in Modern Physics 202, Springer, Berlin, 2004).
15. A. V. Chubukov, D. Pines and J. Schmalian, Conventional and High-T_c Superconductors, in *The Physics of Superconductors*, Vol. 1, eds. K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2003), p. 495.
16. R. B. Laughlin, cond-mat/0209269.
17. B. A. Bernevig, R. B. Laughlin and D. I. Santiago, *Phys. Rev. Lett.* **91**, 147003 (2003).
18. K. Maki, S. Haas, D. Parker, H. Won, B. Dora and A. Virosztek, *Phys. Stat. Sol.* **3**(9), 3156 (2006).
19. J. F. Annett, N. D. Goldenfeld and A. J. Leggett, in *Physical Properties of High Temperature Superconductors V*, ed. D. M. Ginsberg (World Scientific, Singapore, 1996).
20. C. C. Tsuei and J. R. Kirtley, *Rev. Mod. Phys.* **72**, 969 (2000).
21. T. Maier, M. Jarrell, T. Pruschke and J. Keller, *Phys. Rev. Lett.* **85**, 1524 (2000).
22. H. Ott, Conventional and High T_c Superconductors, in *The Physics of Superconductors*, Vol. 1, eds. K. H. Bennemann and J. B. Ketterson (Springer, Berlin, 2003), p. 385.
23. R. Friedberg and T. D. Lee, *Phys. Rev. B* **40**, 6745 (1989).
24. R. Friedberg, T. D. Lee and H.-C. Ren, *Phys. Rev. B* **42**, 4122 (1990).
25. R. Friedberg, T. D. Lee and H.-C. Ren, *Phys. Lett. A* **152**, 417 and 423 (1991).
26. R. Friedberg, T. D. Lee and H.-C. Ren, *Phys. Rev. B* **45**, 10732 (1992).
27. J. Ranninger and S. Robaszkiewicz, *Physica B* **135**, 468 (1985).
28. J. Ranninger, R. Micnas and S. Robaszkiewicz, *Ann. Phys. Fr.* **13**, 455 (1988).
29. A. S. Alexandrov and N. F. Mott, *Rep. Progs. Phys.* **57**, 1197 (1994).
30. D. M. Eagles, *Phys. Rev.* **186**, 456 (1969).
31. R. Micnas, J. Ranninger and S. Robaszkiewicz, *Rev. Mod. Phys.* **62**, 113 (1990).
32. R. Micnas, S. Robaszkiewicz and A. Bussmann-Holder, *Struct. Bond* **114**, 13 (2005).
33. Q. Chen, J. Stajic, Sh. Tan and K. Levin, *Phys. Repts.* **412**, 1 (2005).
34. S. K. Adhikari, M. de Llano, F. J. Sevilla, M. A. Solís and J. J. Valencia, *Physica C* **453**, 37 (2007).

35. M. Botti, E. Cappelluti, C. Grimaldi and L. Pietronero, *Phys. Rev. B* **66**, 054532 (2002).
36. V. V. Tolmachev, *Phys. Lett. A* **266**, 400 (2000).
37. M. de Llano and V. V. Tolmachev, *Physica A* **317**, 546 (2003).
38. J. Batle, M. Casas, M. Fortes, M. de Llano, F. J. Sevilla, M. A. Solís and V. V. Tolmachev, *Cond. Matter Theories* **18**, 111 (2003). Cond-mat/0211456.
39. R. A. Ogg, Jr., *Phys. Rev.* **69**, 243 (1946).
40. V. L. Ginzburg, *Usp. Fiz. Nauk.* **48**, 25 (1952).
41. V. L. Ginzburg, *Fortschr. Phys.* **1**, 101 (1953).
42. J. M. Blatt, *Theory of Superconductivity* (Academic, New York, 1964).
43. M. R. Schafroth, *Phys. Rev.* **96**, 1442 (1954).
44. M. R. Schafroth, S. T. Butler and J. M. Blatt, *Helv. Phys. Acta* **30**, 93 (1957).
45. M. R. Schafroth, *Sol. State Phys.* **10**, 293 (1960).
46. J. M. Blatt and T. Matsubara, *Prog. Theor. Phys.* **20**, 553 (1958).
47. J. M. Blatt and T. Matsubara, *Prog. Theor. Phys.* **23**, 447 and 451 (1960).
48. J. M. Blatt and T. Matsubara, *Prog. Theor. Phys.* **27**, 1137 (1962).
49. N. N. Bogoliubov, *JETP* **34**, 41 (1958).
50. N. N. Bogoliubov, V. V. Tolmachev and D. V. Shirkov, *Fortschr. Phys.* **6**, 605 (1958).
51. N. N. Bogoliubov, V. V. Tolmachev and D. V. Shirkov, *A New Method in the Theory of Superconductivity* (Consultants Bureau, NY, 1959).
52. M. Casas, A. Rigo, M. de Llano, O. Rojo and M. A. Solís, *Phys. Lett. A* **245**, 5 (1998).
53. M. Casas, N. J. Davidson, M. de Llano, T. A. Mamedov, A. Puente, R. M. Quick, A. Rigo and M. A. Solís, *Physica A* **295**, 146 (2001).
54. M. Casas, M. de Llano, A. Puente, A. Rigo and M. A. Solís, *Sol. State Comm.* **123**, 101 (2002).
55. B. S. Deaver, Jr. and W. M. Fairbank, *Phys. Rev. Lett.* **7**, 43 (1961).
56. R. Doll and M. Näbauer, *Phys. Rev. Lett.* **7**, 51 (1961).
57. C. E. Gough, M. S. Colclough, E. M. Forgan, R. G. Jordan, M. Keene, C. M. Muirhead, I. M. Rae, N. Thomas, J. S. Abell and S. Sutton, *Nature* **326**, 855 (1987).
58. C. W. Schneider, G. Hammerl, G. Logvenov, T. Kopp, J. R. Kirtly, P. J. Hirschfeld and J. Mannhart, *Europhys. Lett.* **68**, 86 (2004).
59. A. A. Aligia, A. P. Kampf and J. Mannhart, *Phys. Rev. Lett.* **94**, 247004 (2005).
60. S. I. Koh, *Physica C* **191**, 167 (1992).
61. S. I. Koh, *Phys. Rev.* **49**, 8983 (1994).
62. P. Tarasewicz, *Eur. Phys. J. B* **41**, 185 (2004).
63. Yu. M. Bunkov, A. S. Chen, D. J. Cousins and H. Godfrin, *Phys. Rev. Lett.* **85**, 3456 (2000).
64. C. Gough, private communication (2004).
65. L. N. Cooper, *Am. J. Phys.* **28**, 91 (1960).
66. M. G. López-Arrieta, M. de Llano and M. A. Solís, *Rev. Mex. Fís.* **47**, 54 (2001).
67. T. Timusk and B. Statt, *Rep. Prog. Phys.* **62**, 61 (1999).
68. M. Kugler, Ø. Fischer, Ch. Renner, S. Ono and Y. Ando, *Phys. Rev. Lett.* **86**, 4911 (2001).
69. T. A. Mamedov and M. de Llano, *Phys. Rev. B* **75**, 104506 (2007).
70. A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinskii, *Methods of Quantum Field in Statistical Physics* (Dover, NY, 1975).
71. J. Gavoret and P. Nozières, *Ann. Phys. (NY)* **28**, 349 (1964).
72. P. C. Hohenberg and P. C. Martin, *Ann. Phys. (NY)* **34**, 291 (1965).
73. M. de Llano, F. J. Sevilla and S. Tapia, *Int. J. Mod. Phys. B* **20**, 2931 (2006).

74. P. Ehrenfest and J. R. Oppenheimer, *Phys. Rev.* **37**, 333 (1931).
75. A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971).
76. P. Fulde and R. A. Ferrell, *Phys. Rev.* **135**, A550 (1964).
77. A. I. Larkin and Y. N. Ovchinnikov, *Sov. Phys. JETP* **20**, 762 (1965).
78. T. Mizushima, K. Machida and M. Ichioka, *Phys. Rev. Lett.* **94**, 060404 (2005).
79. K. Yang, *Phys. Rev. Lett.* **95**, 218903 (2005).
80. R. Hulet, private communication (2005).
81. M. Casas, C. Esebbag, A. Extremera, J. M. Getino, M. de Llano, A. Plastino and H. Rubio, *Phys. Rev. A* **44**, 4915 (1991).
82. V. C. Aguilera-Navarro and M. de Llano, *Rev. Mex. Fís* **40**, 167 (1994).
83. M. de Llano, A. Salazar and M. A. Solís, *Rev. Mex. Fís.* **51**, 626 (2005).
84. P. Gosdzinsky and R. Tarrach, *Am. J. Phys.* **59**, 70 (1991).
85. S. K. Adhikari, M. Casas, A. Puente, A. Rigo, M. Fortes, M. A. Solís, M. de Llano, A. A. Valladares and O. Rojo, *Phys. Rev. B* **62**, 8671 (2000).
86. K. Miyake, *Prog. Theor. Phys.* **69**, 1794 (1983).
87. R. Mendoza, *Master's Thesis* (UNAM, 2007).
88. S. K. Adhikari, M. Casas, A. Puente, A. Rigo, M. Fortes, M. de Llano, M. A. Solís, A. A. Valladares and O. Rojo, *Physica C* **351**, 341 (2001).
89. L. Van Hove, *Physica* **25**, 849 (1959).
90. M. Casas, S. Fujita, M. de Llano, A. Puente, A. Rigo and M. A. Solís, *Physica C* **295**, 93 (1998).
91. I. M. Chapnik, *Phys. Status Solidi B* **123**, K1L3 (1984) and refs. therein.
92. J. Hirsch, *Physica C* **341–348**, 213 (2000) and refs. therein.
93. A. Lanzara, P. V. Bogdanov, X. J. Zhou, S. A. Kellar, D. L. Feng, E. D. Lu, T. Yoshida, H. Eisaki, A. Fujimori, K. Kishio, J.-I. Shimoyama, T. Noda, S. Uchida, Z. Hussain and Z.-X. Shen, *Nature* **412**, 510 (2001).
94. E. E. Salpeter and H. A. Bethe, *Phys. Rev.* **84**, 1232 (1951).
95. P. W. Anderson, *Phys. Rev.* **112**, 1900 (1958).
96. P. W. Higgs, *Phys. Lett.* **12**, 132 (1964).
97. S. V. Traven, *Phys. Rev. Lett.* **73**, 3451 (1994).
98. S. V. Traven, *Phys. Rev.* **51**, 3242 (1995).
99. V. C. Aguilera-Navarro, M. Fortes and M. de Llano, *Sol. St. Comm.* **129**, 577 (2004).
100. M. Fortes, M. A. Solís, M. de Llano and V. V. Tolmachev, *Physica C* **364**, 95 (2001).
101. M. Fortes, S. K. Adhikari, M. de Llano and M. A. Solís, *One-dimensional Cooper pairing via the Bethe-Salpeter Equation*, to be published.
102. S. Schmitt-Rink, C. M. Varma and A. E. Ruckenstein, *Phys. Rev. Lett.* **63**, 445 (1989).
103. S. Sachdev and Z. Wang, *Phys. Rev. B* **43**, 10229 (1991).
104. B. D. Brandow, *Phys. Repts.* **296**, 1 (1998).
105. V. L. Berezinskii, *Sov. Phys. JETP* **34**, 610 (1972).
106. J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**, 1181 (1973).
107. J. M. Kosterlitz, *J. Phys. C* **7**, 1046 (1974).
108. D. Jérome, *Science* **252**, 1509 (1991).
109. J. M. Williams, A. J. Schultz, U. Geiser, K. D. Carlson, A. M. Kini, H. H. Wang, W. K. Kwok, M. H. Whangbo and J. E. Schirber, *Science* **252**, 1501 (1991).
110. H. Hori, *Int. J. Mod Phys. B* **8**, 1 (1994).
111. Z. K. Tang, *et al.*, *Science* **292**, 2462 (2001).
112. D. Vollkardt and P. Wölfle, *The Superfluid Phases of Helium 3* (Taylor & Francis, London, 1990).

113. M. Greiner, C. A. Regal and D. S. Jin, *Nature* **426**, 537 (2003).
114. M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, S. Gupta, Z. Hadzibabic and W. Ketterle, *Phys. Rev. Lett.* **91**, 250401 (2003).
115. H. A. Bethe and J. Goldstone, *Proc. Roy. Soc. (London)* **A238**, 551 (1957).
116. L. Belkhir and M. Randeria, *Phys. Rev. B* **49**, 6829 (1994).
117. R. V. Carlson and A. M. Goldman, *Phys. Rev. Lett.* **31**, 880 (1973).
118. R. V. Carlson and A. M. Goldman, *Phys. Rev. Lett.* **34**, 11 (1975).
119. N. N. Bogoliubov, *N. Cim.* **7**, 794 (1958).
120. J. G. Valatin, *N. Cim.* **7**, 843 (1958).
121. G. Rickayzen, *Theory of Superconductivity* (Interscience, NY, 1965).
122. H. Matsui, T. Sato, T. Tkahashi, S.-C. Wang, H.-B. Yang, H. Ding, T. Fujii, T. Watanabe and A. Matsuda, *Phys. Rev. Lett.* **90**, 217002 (2003), see esp. Fig. 3.
123. J. J. Valencia, *Doctoral Thesis* (UNAM, 2007).
124. S. Fujita and S. Godoy, *Quantum Statistical Theory of Superconductivity* (Plenum, NY, 1996), p. 167.
125. P. Samuelsson and M. Büttiker, *Phys. Rev. Lett.* **89**, 046601 (2002).
126. M. J. Holland, S. J. J. M. F. Kokkelmans, M. L. Chiofalo and R. Walser, *Phys. Rev. Lett.* **87**, 1204061 (2001).
127. E. Timmermans, K. Furuya, P. W. Milonni and A. K. Kerman, *Phys. Lett. A* **285**, 228 (2001).
128. M. L. Chiofalo, S. J. J. M. F. Kokkelmans, J. N. Milstein and M. J. Holland, *Phys. Rev. Lett.* **88**, 090402 (2002).
129. Y. Ohashi and A. Griffin, *Phys. Rev. Lett.* **89**, 130402 (2002).
130. A. Rosencwaig, *Phys. Rev. B* **67**, 184514 (2003). See, however, also cond-mat/0305672 and 0410099.
131. J. A. Wilson, *Phil. Mag.* **84**, 2183 (2004) esp. Fig. 2.
132. J. E. Hoffman, K. McElroy, D.-H. Lee, K. M. Lang, H. Eisaki, S. Uchida and J. C. Davis, *Science* **297**, 1148 (2002).
133. K. McElroy, R. W. Simmonds, J. E. Hoffman, D.-H. Lee, J. Orenstein, H. Eisaki, S. Uchida and J. C. Davis, *Nature* **422**, 592 (2003).
134. J. Zaanen, *Nature* **422**, 569 (2003).
135. M. de Llano and J. J. Valencia, *Mod. Phys. Lett. B* **20**, 1067 (2006).
136. J. Guimpel, F. de la Cruz, J. Murduck and I. K. Schuller, *Phys. Rev. B* **35**, 3655 (1987).
137. A.H. Harker, *Superconductivity II* (2002) www.cmmp.ucl.ac.uk/˜ahh/teaching/3C25/Lecture31p.pdf.
138. T. Jacobs, S. Sridhar, Q. Li, G. D. Gu and N. Koshizuka, *Phys. Rev. Lett.* **75**, 4516 (1995).
139. J. E. Sonier, J. H. Brewer, R. F. Kiefl, G. D. Morris, R. I. Miller, D. A. Bonn, J. Cakhalian, R. H. Heffner, W. N. Hardy and R. Liang, *Phys. Rev. Lett.* **83**, 4156 (1999).
140. D. A. Bonn, S. Kamal, K. Zhang, R. Liang, D. J. Baar, E. Klein and W. N. Hardy, *Phys. Rev. B* **50**, 4051 (1994).
141. Z. K. Tang, L.-Y. Zhang, N. Wang, X. X. Zhang, G. H. Wen, G. D. Li, J. N. Wang, C. T. Chan and P. Sheng, *Science* **292**, 2462 (2001).
142. M. Casas, M. de Llano, A. Puente, A. Rigo and M. A. Solís, *Sol. State Comm.* **123**, 101 (2002), see esp. Fig. 1.
143. A. B. Migdal, *JETP* **7**, 996 (1958).
144. J. P. Carbotte, *Revs. Mod. Phys.* **62**, 1027 (1990).
145. Y. H. Kim and P.-H. Hor, *Mod. Phys. Lett. B* **15**, 497 (2001).

146. B. Lorenz, Z. G. Li, T. Honma and P.-H. Hor, *Phys. Rev.* **65**, 144522 (2002).
147. M. de Llano and J. J. Valencia, *Mod. Phys. Lett. B* **20**, 1067 (2006).
148. C. P. Poole, Jr., *et al.*, *Handbook of Superconductivity* (Academic, New York, 2000).
149. A. S. Alexandrov, *Int. J. Mod. Phys. B* **21**, 2301 (2007).
150. Y. J. Uemura, *J. Phys.: Condens. Matter* **16**, S4515 (2004).