Bose-Einstein Condensation in the Relativistic Ideal Bose Gas

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The Bose-Einstein condensation (BEC) critical temperature in a relativistic ideal Bose gas of identical bosons, with and without the antibosons expected to be pair-produced abundantly at sufficiently hot temperatures, is exactly calculated for all boson number densities, all boson point rest masses, and all temperatures. The Helmholtz free energy at the critical BEC temperature is *lower* with antibosons, thus implying that omitting antibosons always leads to the computation of a metastable state.

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Since its theoretical prediction by Einstein in 1925 based on the work in 1924 by Bose on photons, and after languishing for many decades as a mere academic exercise in textbooks, BEC has been observed in the laboratory in laser-cooled, magnetically-trapped ultracold bosonic atomic clouds of $^{87}_{37}$Rb [1], $^{7}_{3}$Li [2], $^{23}_{11}$Na [3], $^{1}_{1}$H [4], $^{85}_{37}$Rb [5], $^{4}_{2}$He [6], $^{41}_{19}$K [7], $^{133}_{55}$Cs [8(a)], $^{174}_{70}$Yb [8(b)], and $^{52}_{24}$Cr [9]. Previously, BEC in a gas of excitons had also been reported [10]. More recently, BEC has been seen as well in fermionic atomic gases of $^{40}_{19}$K [11] and $^{6}_{3}$Li [12] as a result of some of the fermions presumably Cooper-pairing [13] into bosons. The role of *hole* Cooper pairs accounted for or not, along with the usual particle Cooper pairs, has been explored [14–17] with striking implications for many-electron superconducting states within a BEC scenario. Their effects in neutral-fermion many-particle systems are yet to be investigated. It is the analogy of hole pairs with antibosons which has piqued our interest in the problem to be dealt with in this Letter. Although it is hard to imagine a many-boson system whose constituent bosons do not disintegrate into their components at the high temperatures where antiboson production becomes substantial, it may shed light on the dynamics of many-fermion systems where both particle- and hole-Cooper pairing can occur at *all* temperatures. Moreover, recent specific-heat measurements [18] in $TlCuCl_3$ suggest a magnon BEC with a relativistic dispersion relation.

In early papers [19–21] on the relativistic ideal boson gas (RIBG), explicit BEC critical transition temperature T_c-formulae were derived for both the nonrelativistic and ultrarelativistic limits, and specific-heat anomalies at T_c were studied. In addition, Refs. [20,21] considered all space dimensions $d > 0$ and delved into the relation between d and various critical exponents. Antiboson production, however, was *not* accounted for. The first papers to include *both* bosons and antibosons appear to be Refs. [22,23] where high-temperature expansions for the various thermodynamic functions (pressure, particle-number-density, entropy, specific heats, etc.) were derived. Extensive numerical work in d dimensions that does not rely such high-temperature expansions was reported in Refs. [24,25]. In an elegant treatment [26] with inverse Mellin transforms, the specific-heat anomaly of the RIBG at its BEC T_c was found to be washed out when pair-production was included. The relationship between the BEC of the RIBG and spontaneous-symmetry breaking was explored in Refs. [23,27]; see also the rather complete Ref. [28], esp. Sec. 2.4. The so-called BCS [29]-to-Bose crossover scenario (see Ref. [17] and refs. therein), and even the pseudogap concept [30] of superconductors, first seemed to have appeared in quark physics in Ref. [31]; for a review see Ref. [32]. More recently, *two* "crossovers" have been identified [33] in an interacting *fermion* gas where pairing into bosonic Cooper pairs [13] can occur to form a relativistic superfluid as an example of the BCS-Bose crossover followed by a Bose-to-RIBG/BEC-crossover where both antibosons as well as bosons dominate the thermodynamics. A fully relativistic detailed study [34] of these crossovers at zero temperature has also appeared.

In this Letter, we exhibit, as a function of boson number density, exact BEC transition temperatures for the RIBG gas of identical bosons with and without antibosons in 3D. The system with both kinds of bosons always has the higher T_c, i.e., is the system with the first BEC singularity that appears as it is cooled. This suggests that the Helmholtz free energy might be lower and thus correspond thermodynamically to the *stable* system as opposed to a *metastable* system for the lower-T_c system. It is then calculated and indeed found to be lower at all densities for the complete problem with both bosons and antibosons, when compared to the problem without antibosons. This implies that the omission of antibosons will *not* lead to stable states.

The number of bosons N of mass m that make up an ideal boson gas in d dimensions (*without* antibosons) is $N = \sum_{\mathbf{k}} n_{\mathbf{k}} \equiv \sum_{\mathbf{k}} [e^{\beta\{|E_k| - \mu(T)\}} - 1]^{-1}$ where $\beta = 1/k_B T$, k_B is the Boltzmann constant, and $\mu(T)$ is the boson chemical potential. Here, the total energy of each boson is

$$|E_k| \equiv \sqrt{c^2\hbar^2k^2 + m^2c^4} \quad (1)$$

$$= mc^2 + \hbar^2k^2/2m + O(k^4) \quad \text{if } c\hbar k \ll mc^2 \quad \textbf{NR} \quad (2)$$

$$= c\hbar k\left[1 + \frac{1}{2}(mc/\hbar k)^2 + O(k^{-4})\right] \quad \text{if } c\hbar k \gg mc^2 \quad \textbf{UR} \quad (3)$$

where k is the boson wave number, m its rest mass, and c is the speed of light. The two limits refer to the nonrelativistic (NR) and ultrarelativistic (UR) extremes. For a cubic box of side length L in the continuous limit, the sums over the d-dimensional wave vector $\mathbf{k}$ become integrals as $\sum_{\mathbf{k}} \to (L/2\pi)^d \int d^dk$. At the BEC critical transition temperature T_c, $\mu(T_c) = mc^2$ and the boson number density can be expressed as

$$n \equiv \frac{N}{L^d} = \frac{1}{(2\pi)^d}\int d^dk \frac{1}{\exp[\beta_c(|E_k| - mc^2)] - 1} \quad (4)$$

where $\beta_c \equiv 1/k_BT_c$. In the nonrelativistic (NR) extreme, (2) inserted into (4) leaves $n = (2\pi)^{-d}\times \int d^dk\{\exp[\beta_c(\hbar^2k^2/2m)] - 1\}^{-1}$. Putting $d^dk = [2\pi^{d/2}/\Gamma(d/2)]k^{d-1}dk$ when integrating over terms independent of angles gives an expression for n as function of T_c in terms of the Bose function [35] $g_\sigma(z)$ of $z \equiv \exp(\mu/k_BT_c)$ which for $z = 1$ diverges, namely $g_\sigma(1) \to \infty$ when $\sigma \leq 1$, but becomes the Riemann Zeta function $\zeta(\sigma) < \infty$ when $\sigma > 1$. Here, $\Gamma(\sigma)$ is the gamma function. Solving for the critical temperature then gives

$$k_BT_c^{\text{NR-}B} = \frac{2\pi\hbar^2}{m}[n/\zeta(d/2)]^{2/d} \quad (5)$$

where the superscript NR-B stands for the nonrelativistic limit with bosons (B) but no antibosons ($\bar{B}$). In 3D, this reduces to the familiar textbook result $k_BT_c^{\text{NR-}B} \simeq 3.31\hbar^2n^{2/3}/m$ since $\zeta(3/2) \simeq 2.612$. In the ultrarelativistic (UR) extreme, the leading term of (3) inserted into (4) leads to $T_c = 0$ for all $d \leq 1$ since then $g_d(1)$ diverges. However, for $d > 1$,

$$k_BT_c^{\text{UR-}B} = \left[\frac{\hbar^dc^d2^{d-1}\pi^{d/2}\Gamma(d/2)}{\Gamma(d)\zeta(d)}\right]^{1/d} n^{1/d} \quad (6)$$

which in 3D becomes $k_BT_c^{\text{UR-}B} = \hbar c\pi^{2/3}[n/\zeta(3)]^{1/3} \simeq 2.017\hbar cn^{1/3}$ as $\zeta(3) \simeq 1.20206$. In 2D, $T_c^{\text{UR-}B} \neq 0$ unlike the common instance with quadratic dispersion where T_c vanishes because $g_1(1)$ diverges; specifically, $k_BT_c^{\text{UR-}B} = \hbar c[2\pi n/\zeta(2)]^{1/2} \simeq 1.954\hbar cn^{1/2}$ since $\zeta(2) = \pi^2/6$.

At sufficiently high temperatures such that $k_BT \gg mc^2$, boson-antiboson pair production occurs abundantly; this has been stressed by Huang [36]. The total energy E_k of each particle always satisfies $E_k^2 = c^2\hbar^2k^2 + m^2c^4$ so that $E_k = \pm\mid E_k\mid$ where $\mid E_k\mid$ is given by (1) and with the $+$ sign referring to bosons and the $-$ sign to antibosons. Instead of $N = \sum_{\mathbf{k}} n_{\mathbf{k}}$, the *complete* number equation is now [22]

$$N - \bar{N} \equiv \sum_{\mathbf{k}}(n_{\mathbf{k}} - \bar{n}_{\mathbf{k}}) = \sum_{\mathbf{k}}\left[\frac{1}{\exp[\beta(|E_k| - \mu)] - 1} - \frac{1}{\exp[\beta(|E_k| + \mu)] - 1}\right] \quad (7)$$

where $n_{\mathbf{k}}$ ($\bar{n}_{\mathbf{k}}$) is the average number of bosons (antibosons) in the state of energy $\pm|E_k|$, respectively, at a given temperature T and N ($\bar{N}$) is their respective total number at that temperature. Since $n_{\mathbf{k}}, \bar{n}_{\mathbf{k}} > 0$ for all $\mathbf{k}$ and $E_0 = mc^2$, the chemical potential must be bounded by $-mc^2 \leq \mu \leq mc^2$. Instead of N constant, one must now impose the constancy of $N - \bar{N}$ to extract the correct BEC critical temperature, say, $T_c^{B\bar{B}}$ referring to both bosons (B) and antibosons ($\bar{B}$). Since $|\mu(T_c^{B\bar{B}})| = mc^2$, (7) becomes

$$n \equiv (N - \bar{N})/L^d = \frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d}\int_0^\infty dkk^{d-1} \times \frac{\sinh(\beta_cmc^2)}{\cosh(\beta_c\sqrt{c^2\hbar^2k^2 + m^2c^4}) - \cosh(\beta_cmc^2)}. \quad (8)$$

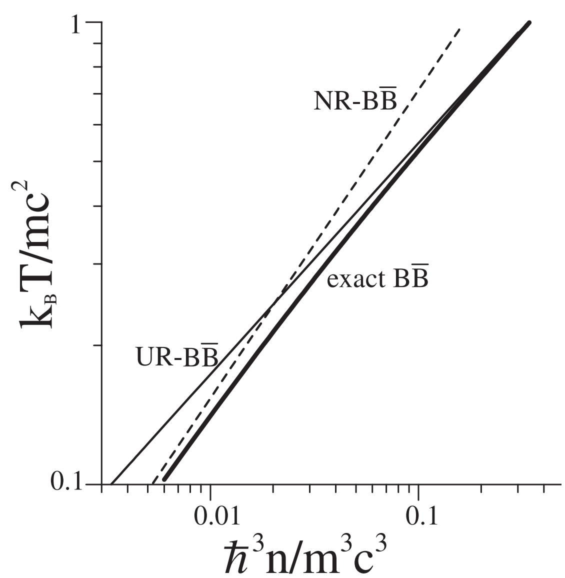

FIG. 1. BEC T_cs (in units mc^2/k_B) as function of boson number-density n expressed in dimensionless form as $\hbar^3n/m^3c^3$. Thick curve labeled "exact $B\bar{B}$" is exact numerical result of (8) that corresponds to BEC T_c in a RIBG with both bosons B *and* antibosons $\bar{B}$. Thin full straight line labeled "UR-$B\bar{B}$" is the ultrarelativistic limit (9) for $d = 3$ with both kinds of bosons. Dashed straight line labeled "NR-$B\bar{B}$" is its corresponding nonrelativistic limit (5) for $d = 3$ and tends asymptotically to the "exact $B\bar{B}$" curve at smaller n.

This is an exact expression for the BEC T_c of an ideal Bose gas at *any* temperature as it includes both bosons and antibosons; it is consistent with Eq. (13) of Ref. [24].

At low enough temperatures such that $k_B T_c \ll mc^2$, antibosons can be neglected and (8) simplifies to, say, $T_c^{\text{NR-}B\bar{B}}$ which is precisely (5) as expected. In the opposite extreme, $k_B T_c \gg mc^2$ (8) leads to the limiting expression, say,

$$k_B T_c^{\text{UR-}B\bar{B}} = \left[\frac{\hbar^d c^{d-2}\Gamma(d/2)(2\pi)^d}{4m\pi^{d/2}\Gamma(d)\zeta(d-1)}\right]^{1/(d-1)} n^{1/(d-1)} \quad (9)$$

that sharply differs from (6). In 3D, this becomes $k_B T_c^{\text{UR-}B\bar{B}} = (3\hbar^3 c/m)^{1/2} n^{1/2}$, a result apparently first reported in Ref. [22]. This novel relation has suggested [18] itself experimentally as a magnon BEC in specific-heat measurements in TlCuCl_3.

As functions of the dimensionless boson number density $\hbar^3 n/m^3 c^3$, Fig. 1 displays the behavior of the exact $T_c^{B\bar{B}}$ (in units of mc^2/k_B) numerically extracted from (8) for $d = 3$ (thick full curve labeled "exact $B\bar{B}$") compared with the nonrelativistic limit $T_c^{\text{NR-}B\bar{B}}$ from (5) (dashed line labeled "NR-$B\bar{B}$") and with the ultrarelativistic $T_c^{\text{UR-}B\bar{B}}$ just stated (full thin line labeled "UR-$B\bar{B}$"). Figure 2 shows how, at sufficiently high densities n and/or sufficiently small boson rest mass m, the exact $T_c^{B\bar{B}}$ (again in units of mc^2/k_B, full curve labeled "exact $B\bar{B}$") is clearly the *first* BEC singularity encountered as the many-boson system is cooled, compared with the "later" BEC singularity in the system *without* antibosons at T_c^B (dashed curve labeled "exact B") extracted numerically from (4). It is then tempting to speculate that the boson gas with *both* kinds of bosons will be the more stable, i.e., have a *lower* Helmholtz free energy at all critical temperatures, at any fixed $\hbar^3 n/m^3 c^3$. This will now be shown to be the case indeed.

The exact Helmholtz free energy per unit volume $V = L^3$ for the boson-antiboson 3D mixture, when $T = T_c \equiv 1/k_B\beta_c$ and $\mu = mc^2$, is

$$F^{\text{exact}B\bar{B}}(T_c, V)/V = nmc^2 + (k_B T_c/2\pi^2)\int_0^\infty dk k^2\{\ln[1-\exp(\beta_c[mc^2 - |E_k|])] + \ln[1-\exp(-\beta_c[mc^2 + |E_k|])]\} \quad (10)$$

where $|E_k|$ is given by (1) and $n = (N - \bar{N})/V$, and T_c is extracted numerically from (8) for each value of n. If $k_B T_c \ll mc^2$, antibosons can be neglected entirely and $F^{\text{exact}B}(T_c, V)$ is just (8) but without the second log term and for which T_c is now extracted numerically from (4) instead of from (8) for each value of n. Figure 3 is the difference between the two free energies and more clearly shows why $F^{\text{exact}B\bar{B}}(T_c, V)$ is always lower. The inset figure shows the behavior of these two free energies, each evaluated at their appropriate T_c value; by inspection, both curves correspond as they should to *positive* pressures P since $P = -(\partial F/\partial V)_{N,T}$. Figure 3 proves the speculation advanced.

In summary, based on exact numerical calculations of BEC T_cs in a 3D RIBG with and without the antibosons expected to be pair produced at higher and higher tempera-

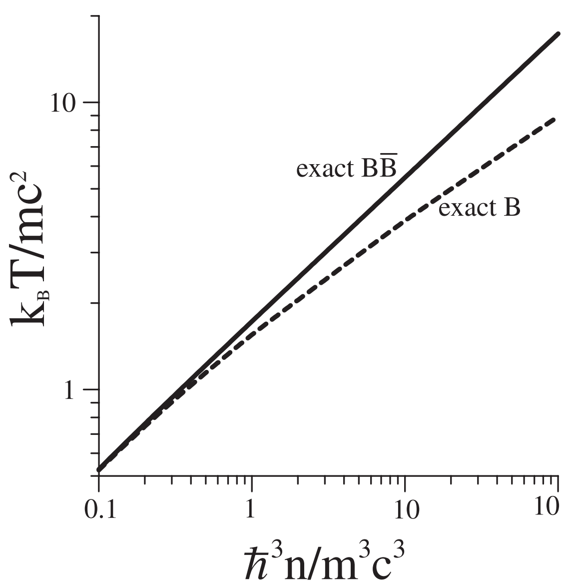

FIG. 2. Same as Fig. 1 comparing exact $B\bar{B}$ RIBG T_c extracted from (8) against that of exact B from (4).

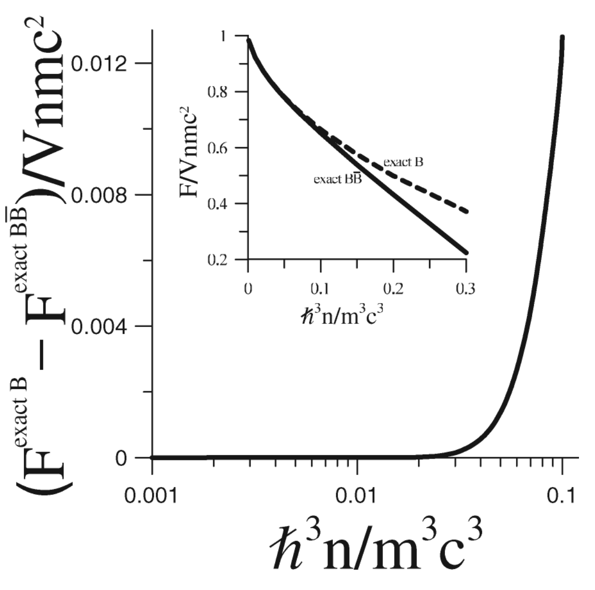

FIG. 3. Difference between exact Helmholtz free energy density F/V (in units of nmc^2 which is the total rest-mass energy density) (full curve labeled "exact $B\bar{B}$") and that without antibosons (dashed curve labeled "exact B") using same horizontal axes as in Figs. 1 and 2. The inset figure displays the two free energies.

tures, the highest critical T_c is that associated with the RIBG system with *both* kinds of bosons taken into account. At lower n and/or larger m, the higher T_c merges smoothly from above onto the lower T_c of the RIBG system with antibosons neglected. Comparing the associated Helmholtz free energies shows that the RIBG with both kinds of bosons has *lower* values for all n and m and thus substantiates the initial suspicion that the RIBG system with no antibosons is metastable with respect to the one with both kinds of bosons.

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