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INTRIGUING ROLE OF HOLE-COOPER-PAIRS IN SUPERCONDUCTORS AND SUPERFLUIDS

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The role in superconductors of *hole*-Cooper-pairs (CPs) are examined and contrasted with the more familiar *electron*-CPs, with special emphasis on their "background" effect in enhancing superconducting transition temperatures T_c — even when *electron*-CPs drive the transition. Both kinds of CPs are, of course, present at *all* temperatures. An analogy is drawn between the hole CPs in any many-fermion system with the antibosons in a relativistic ideal Bose gas that appear in substantial numbers *only* at higher and higher temperatures. Their indispensable role in yielding a lower Helmholtz free energy equilibrium state is established. For superconductors, the problem is viewed in terms of a generalized Bose-Einstein condensation (GBEC) theory that is an extension of the Friedberg-T.D. Lee 1989 boson-fermion BEC theory of high-T_c superconductors in that the GBEC theory includes hole CPs as well as electron-CPs — thereby containing as well as further extending BCS theory to higher temperatures with the same weak-coupling electron-phonon interaction parameters. We show that the Helmholtz free energy of both 2e- and 2h-CP pure condensates has a positive second derivative, and are thus stable equilibrium states. Finally, it is conjectured that the role of hole pairs in ultra-cold fermionic atom gases will likely be negligible because the very low densities involved imply a "shallow" Fermi sea.

Keywords: Superconductivity; fermionic atom superfluidity; hole-Cooper-pairs; Bose-Einstein condensation; electron-phonon coupling.

1. Introduction

Quantum many-body matter comprises a wide variety of both mass ρ and number densities n. For superconductors, superfluids and Bose-Einstein condensates (BECs), a wide range of critical temperatures T_c is also present. For many-fermion systems such as superconductors one has $n = 10^{19} - 10^{23}\ cm^{-3}$ which implies Fermi temperatures $T_F = 70 - 10^5 K$. For neutral-fermion superfluids, such as i) liquid-^3He one has $n \simeq 10^{22}\ cm^{-3}$ (or $\rho \simeq 0.1\ g/cm^3$, i.e., about a 1/10th that of water) and $T_F = 3 - 5K$; ii) neutron-star matter $\rho = 10^{15}\ g/cm^3$ and $T_F = 10^{12} K$; iii) dilute mixtures of ^3He in ^4He; and iv) ultra-cold fermionic BECs with enormous average spacings (when compared with inter-atomic potential ranges) $\rho \simeq 10^{-8}\ g/cm^3$ and $T_F < 10^{-6} K$.

Here, we survey the role of hole-Cooper-pairs in many-fermion gases, starting with their analogy with antiparticles such as antineutrinos and antibosons in general.

2. Role of Antiparticles

2.1. *In high-energy physics*

It is well-known that starting from the Einstein equation $E = m_{rel}c^2$ relating the energy E of a particle with relativistic mass $m_{rel} \equiv m/\sqrt{1 - v^2/c^2}$ where m is the particle rest mass, v its velocity and c the speed of light, as well as the relativistic expression for linear momentum $p = m_{rel}v$, one can readily deduce the relativistic energy-momentum relation $E^2 = p^2c^2 + m^2c^4$ for the particle. Solving for the energy E leads to the *two* solutions $E = \pm(\hbar^2K^2c^2 + m^2c^4)^{1/2} \equiv \pm \mid E_K \mid$. In 1928, Dirac decided *not to exclude* the minus-sign solution and discovered antiparticles, the first example of which was the antielectron or positron that was observed by Anderson in 1932. The simplest instance of the occurrence of antimatter in high-energy physics is perhaps beta decay whereby an isolated neutron n decays into a proton p, an electron e and an *anti*neutrino $\overline{\nu}_e$, and occurs with a half-life of about 15 minutes, namely

$$n \rightarrow p + e + \overline{\nu}_e. \tag{1}$$

Here, conservation of lepton number *requires* that the neutrino on the right-hand side be an antiparticle. This contrasts with the inverse process

$$p + e \rightarrow n + \nu_e \tag{2}$$

known as electron capture as occurs, e.g., under the influence of the enormous gravitational attractions in the evolution of an ideal white dwarf into a neutron star.

2.2. *In relativistic BEC*

For *any* temperature T the correct energy-momentum relation of identical bosons of mass m in a many-boson gas is, of course, not strictly $E_K = \hbar^2K^2/2m$ but rather

$$|E_K| \equiv \sqrt{c^2\hbar^2K^2 + m^2c^4} = mc^2 + \hbar^2K^2/2m + O(K^4) \ if \ c\hbar K \ll mc^2 \qquad \mathbf{NR}$$

$$= c\hbar K[1 + \frac{1}{2}(mc/\hbar K)^2 + O(K^{-4})] \ if \ c\hbar K \gg mc^2 \quad \mathbf{UR}$$

where NR and UR stand for the nonrelativistic and ultrarelativistic limits. In the UR case, the BEC singularity at T_c^{Nconst} that follows by imposing constancy of the total number of bosons N given by

$$N \equiv N_0(T) + \sum_{\mathbf{K}\neq\mathbf{0}} [\exp\beta(|\, E_K \,| - \mu) - 1]^{-1} \tag{3}$$

is well-known to be $T_c^{Nconst} = (\hbar c/k_B)[\pi^2 n/\zeta(3)]^{1/3}$ where $n \equiv N/L^3$ with L the system size and $\zeta(3) \simeq 1.20206$ is the Riemann Zeta function of order 3. Not surprisingly, there is no mass m dependence. On the other hand, in the NR case one gets the familiar formula $T_c^{Nconst} \simeq 3.31\hbar^2 n^{2/3}/m$ which *does* have a mass dependence. Indeed, both these expressions for T_c^{Nconst} are special cases of a general dispersion relation $E_K = C_s K^s$, with C_s a constant, in any dimensionality d and implying the general T_c-formula[1]

$$k_B T_c^{Nconst} = C_s \left[\frac{s\Gamma(d/2)(2\pi)^d n}{2\pi^{d/2}\Gamma(d/s)g_{d/s}(1)}\right]^{s/d} \tag{4}$$

where $\Gamma(x)$ is the gamma function and $g_\sigma(1) \equiv \zeta(\sigma)$ if $\sigma > 1$ but ∞ if $\sigma \leq 1$. This latter property leads to the well-known conclusion that $T_c \equiv 0 \ \forall \ d \leq s$ but is otherwise nonzero.

These results, however, ignore the appearance at higher and higher temperatures of substantial numbers of antibosons. At high T's BEC *must account for, say,* $\overline{N}'$ *antibosons* along with N' bosons. Here, $\overline{N}'$ is N' as in (3) but with $+\mu$ instead of $-\mu$. Thus, in general it is not N' that is constant but rather $N' - \overline{N}'$. The latter is given by

$$(N' - \overline{N}') \equiv N \equiv nL^d \equiv \sum_{\mathbf{K}} (n_{\mathbf{K}} - \overline{n_{\mathbf{K}}}) = \sum_{\mathbf{K}} \{[\exp\{\beta(|\, E_K \,| - \mu)\} - 1]^{-1} \tag{5}$$
$$- same \ with \ \mu \rightarrow -\mu\}.$$

In the UR limit of this *exact* equation one can extract $T_c^{N'-\overline{N}'const} = (\hbar^{3/2}c^{1/2}/k_B)(3n/m)^{1/2}$ which, surprisingly, now depends on m. The *exact* value of the BEC T_c associated with (5) was found[2] numerically to be *higher* than that associated with (3) for *all* values of the dimensionless number density $\hbar^3 n/m^3c^3$. Their associated Helmholtz free energies were similarly found to be *lower* when antibosons are not neglected as in (5) than when only bosons are included as in (3). This means that the complete system with bosons as well as antibosons is the stable thermodynamic state whereas the system without antibosons is a metastable state.

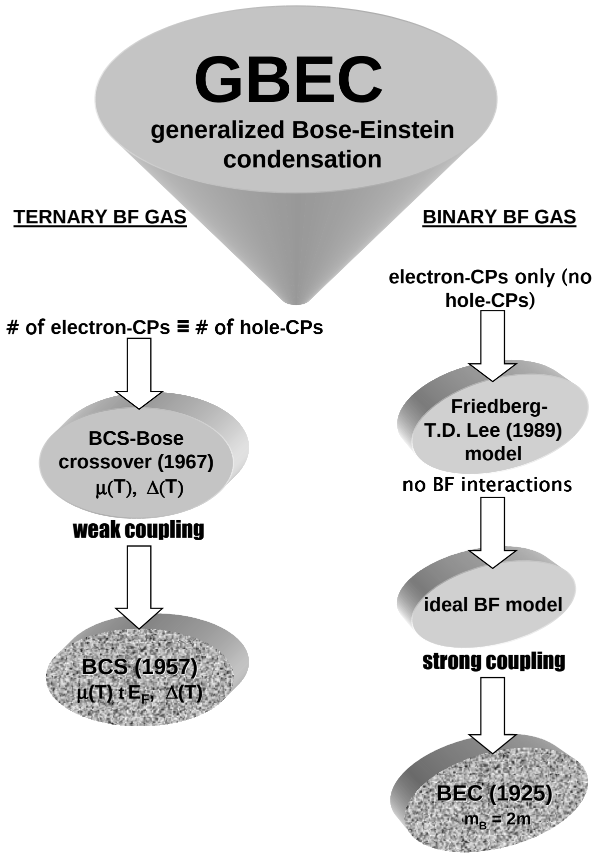

Fig. 1. Flowchart showing how the GBEC theory reduces to: i) the BCS-Bose crossover theory [defined by two coupled transcendental equations for $\Delta(T)$ and $\mu(T)$] whenever the number of 2e-CPs equals the number of 2h-CPs in *both* condensate $n_0(T) = m_0(T)$ *and* noncondensate $n_{B+}(T) = m_{B+}(T)$; ii) the latter reduces to the ordinary BCS theory [defined by the gap equation for $\Delta(T)$] whenever interelectronic coupling λ is sufficiently small so that $\mu(T) \simeq E_F$, the Fermi energy; iii) the three theories shown on the rhs leg correspond to entirely ignoring 2h-CPs and includes the ordinary BEC theory when *all* electrons are paired into bosons of mass $2m$ and number density $n/2$ (i.e., strong coupling).

3. Hole-Cooper-Pairs in a Generalized BEC Theory

A generalized BEC theory (GBEC) of a *ternary* gas mixture — composed of two-electron (2e)[3] Cooper pairs (CPs), two-hole (2h) CPs and unpaired electrons — was described earlier.[4–6] It leads to three coupled transcendental equations for three unknown functions of absolute temperature T which are $n_0(T)$ (the condensate number density of 2e-CPs), $m_0(T)$ (the same for 2h-CPs) and the electron chemical potential $\mu(T)$.

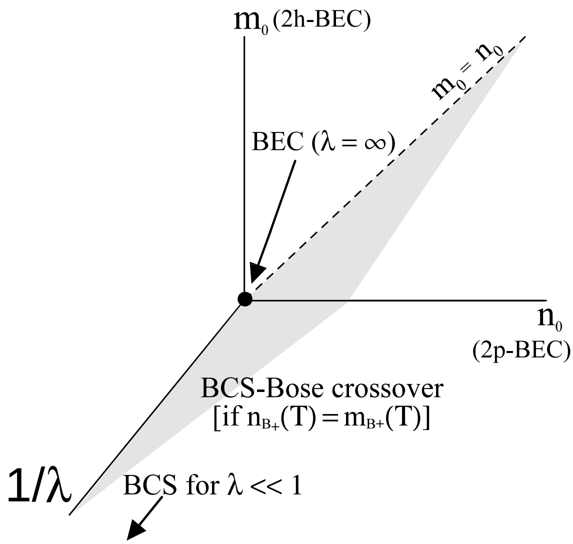

Fig. 2. Parameter octant defined by the two condensate densities $n_0(T) \geqslant 0$ and $m_0(T) \geqslant 0$ as well as the (also nonnegative) *inverse* $1/\lambda \geqslant 0$ of the interelectronic coupling λ, and valid in principle at all temperatures T. The GBEC describes a ternary gas and holds in the *entire* octant. The BCS-Bose crossover theory occurs *only* on the shaded plane defined by $n_0(T) \equiv m_0(T)$ provided the additional restriction $n_{B+}(T) = m_{B+}(T)$ is imposed whereby the total number of 2p noncondensate CPs equals that of 2h CPs. Here 2p refers to two-particle, i.e., two electrons in superconductivity and two fermionic atoms (as opposed to two fermionic holes) in fermionic superfluidity. BCS theory is valid along the forefront where $\lambda \ll 1$ of the shaded BCS-Bose crossover plane. For quadratically-dispersive bosons the usual BEC theory ensues at the origin of the octant where $m_0(T) = 0$ for all T and $n_0(T_c) = 0$, giving there the *implicit* expression $T_c \simeq 3.31\hbar^2 n_B(T_c)^{2/3}/2mk_B$. This result has the same *form* as the standard *explicit* BEC T_c-formula for mass $2m$ bosons and where the boson number density n_B is, of course, independent of T_c.

The original BCS-Bose crossover picture for the electronic gap $\Delta(T)$ and chemical potential $\mu(T)$ is now supplemented by the central relation

$$\Delta(T) = f\sqrt{n_0(T)} = f\sqrt{m_0(T)} \tag{6}$$

where f is a boson-fermion vertex interaction coupling constant inherent to the GBEC theory. All three functions $\Delta(T)$, $n_0(T)$ and $m_0(T)$ have the familiar "half-bell-shaped" forms. Namely, they are zero above a certain critical temperature T_c, and rise monotonically upon cooling (lowering T) to maximum values $\Delta(0)$, $n_0(0)$ and $m_0(0)$ at $T = 0$. The energy gap $\Delta(T)$ is the order parameter describing the superconducting (SC) [or superfluid (SF)] condensed state, while $n_0(T)$ and $m_0(T)$ are the BEC order parameters depicting the macroscopic occupation that occurs below T_c in a BE condensate. This $\Delta(T)$ is precisely the BCS energy gap if the GBEC theory coupling f is made to correspond to $\sqrt{2V\hbar\omega_D}$ where V and $\hbar\omega_D$

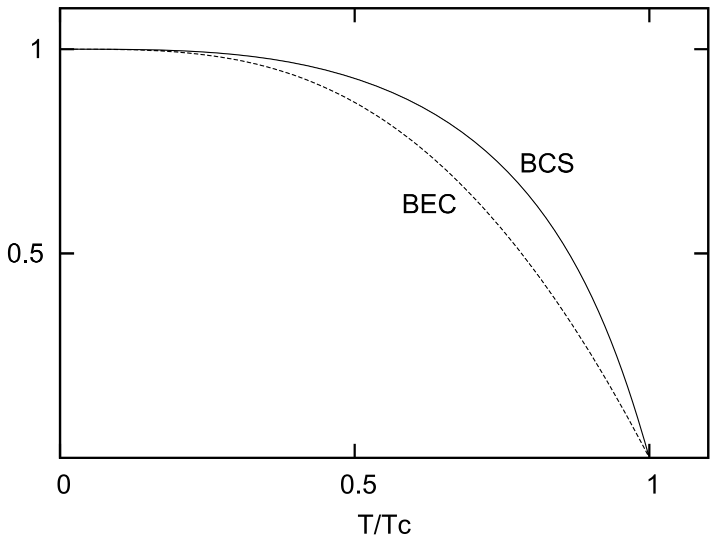

Fig. 3. Order parameters (normalized to unity on both axes) of BCS theory and of the BEC theory as related together by the GBEC theory in accordance with (7).

are the two parameters of the BCS model interelectronic interaction. Evidently, the BCS and BEC T_cs are the same. Writing (6) for $T = 0$ and dividing this into (6) gives the much simpler f-*independent* relation involving order parameters, as well as temperatures T, *normalized* to unity in the interval $[0, 1]$, namely

$$\Delta(T)/\Delta(0) = \sqrt{n_0(T)/n_0(0)} = \sqrt{m_0(T)/m_0(0)} \xrightarrow[T\to 0]{} 1 \text{ as well as} \xrightarrow[T\geq T_c]{} 0. \quad (7)$$

The first equality, apparently first obtained in Ref. 8, connects in a simple way the two heretofore unrelated "half-bell-shaped" order parameters of the BCS and the BEC theories. The second equality[4−6] implies that a BCS condensate is precisely a BE condensate of equal numbers of 2e- and 2h-CPs. Figure 3 shows the relationship between the normalized order parameters of BCS and BEC. Since (7) is *independent* of the particular two-fermion dynamics of the problem, it can be expected to hold for either SCs and SFs.

The insensitivity of T_c/T_F exhibited in Fig. 3 as to whether BF vertex interactions are present ($f \neq 0$) or not ($f = 0$) is a striking result that suggests how "good" the zeroeth-order Hamiltonian H_0 chosen in the GBEC Ref. 6, Eq. (1), and how "unimportant" the interaction Hamiltonian H_{int} (which vanishes as $f \to 0$) Ref. 6, Eq. (2), turn out to be—at least in determining critical temperatures.

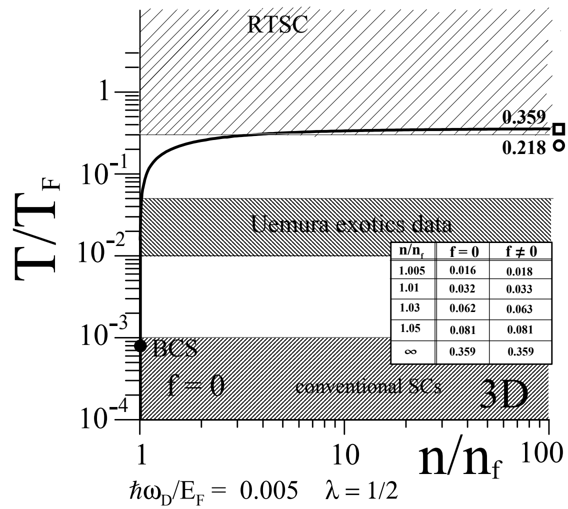

Fig. 4. Phase boundary curve showing net two- to three-order-of-magnitude enhancement, over the BCS prediction of T_c (in units of T_F) for the BCS model interaction with $\lambda = 1/2$ and $\Theta_D/T_F = 0.005$, obtained with the GBEC theory with $f = 0$ for the 2e-CP GBE condensate. Square symbol on the extreme right give $n/n_f \to \infty$ limit of 2e-CP GBE condensate T_c/T_F value (third column of Table 1). The open circle marks the familiar value 0.218 (last column of Table 1) associated with an ideal Fermi gas in which all fermions are pair into bosons of mass $2m$ and number density $n/2$; it is given for reference. RTSC stands for room-temperature superconductivity in a material with $T_F = 10^3 K$. Inset highlights small differences in T_c/T_F when calculations are repeated with $f \neq 0$.

Table 1. GBEC 2e-CP condensate critical temperatures T_c/T_F when $n/n_f \to \infty$. The BCS value (dot in Fig. 4) is given by $T_c/T_F = 1.134(\Theta_D/T_F)\exp(-1/\lambda) \simeq 0.0008$ for $\lambda = 1/2$ and $\Theta_D/T_F = 0.005$ and occurs as a special case of the GBEC phase diagram at $n/n_f = 1$. Note enhancements of T_c/T_F above the familiar value of 0.218 (in bold) corresponding to a many-electron system with all electrons paired of asterisked $(*)$ values associated with a hole-CP background present, i.e., $m_{B+}(T) \neq 0$, in the 2e-CP condensate specified within the GBEC by the condition that the 2h-CP number density in $K = 0$ state $m_0(T) \equiv 0$ for all T. Here n is the electron number density, $n_f(T)$ the *unpaired*-electron number density, $\varepsilon_K = C_s K^s$ the bosonic CP dispersion ($s = 1$ or 2 signifying linear or quadratic), $m_{B+}(T)$ the 2h-CP number density in *all* $K > 0$ states, Δ the zero-temperature fermionic gap and 2Δ the zero-temperature bosonic CP gap in the linearly-dispersive $\varepsilon_K = 2\Delta + \lambda\hbar v_F K/4$ Ref. 9 bosons moving in the Fermi sea, while $s = 2$ refers to quadratically-dispersive $\varepsilon_K = \hbar^2 K^2/4m$ bosons (as, e.g., in Refs. 10 and 11) and is obviously independent of coupling λ. Uemura exotics data on both 2D and 3D SCs is taken from Ref. 12 while the lowest shaded area refers to conventional 3D SCs.

3D ($\lambda = 1/2$ and $\Theta_D/T_F = 0.005$)	$s = 1,\ 2\Delta = 0$	$s = 1,\ 2\Delta \neq 0$	$s = 2$
with $n_f(T) = 0$ (i.e., all e's paired)	0.129	0.130	**0.218**
with $m_B(T) \equiv m_0(T) + m_{B+}(T) = 0\ \forall\ T$	0.127	0.127	0.204
with $m_0(T) = 0$ but $m_{B+}(T) \neq 0\ \forall\ T$	0.359	0.361*	0.507*

4. Cold-atom BECs, Bosonic and Fermionic

Based on Ref. 13, p. 21, with additions and modifications, Table 2 reveals ranges over 12 orders of magnitude in number density (in particles cm^3) for several physical systems. Also given, where appropriate, are condensation temperatures T_c. The very-low-density conjectured "Efimov liquid" is based on Ref. 14.

Table 2. Number densities in particles/cm^3 for a diverse variety of many-body systems. Also given where appropriate are the critical temperatures T_c (in K) which refer to superconductor or superfluidity or cold-atom BEC transition temperatures.

Many-body system	**statistics**	**number density** (cm^{-3})	T_c (K)
electron gas in metals	FD	$10^{22} - 10^{23}$	$0 - 23$
liquid ^{4}He	BE	$\sim 10^{22}$	2.2
liquid ^{3}He	FD	$\sim 10^{22}$	2×10^3
exotic SCs (including cuprates)	FD	$10^{21} - 10^{22}$	$1 - 164$
Air (STP) [78% N_2 + 21% O_2 $+\cdots$]	-	$\sim 10^{19}$	-
ultracold Bose gases	BE	$10^{12} - 10^{15}$	$10^{-8} - 10^{-5}$
ultracold Fermi gases	FD	$10^{12} - 10^{13}$	$10^{-7} - 10^{-6}$
conjectured "Efimov liquid"	BE or FD	$\sim 10^{10}$	-

Below are compiled empirical parameters associated with both bosonic and fermionic ultra-cold gases where BEC has been observed. Along with the usual four states of matter—gas, liquid, solid and plasma—they constitute what have sometimes been termed the Vth and VIth states.

Table 3. Ultra-cold bosonic-atom BEC (sometimes dubbed the "*Vth state of matter*") experimental parameters associated with trapped bosonic gases in which BEC has been observed to date, N and N_0 being the number of atoms in the initial cloud and in the condensate, respectively; T_c the BEC transition temperature; n_0 the reported boson (or peak atom) number density at T_c of the condensate in cm^{-3}; $n_0^{-1/3}$ is average interbosonic spacing in $\mathring{A}$.

BOSONS	$^{87}_{37}$Rb	$^{7}_{3}$Li	$^{23}_{11}$Na	$^{1}_{1}$H	$^{85}_{37}$Rb
Year/Ref.	1995[15]	1995[16]	1995[17]	1998[18]	2000[19]
N	4×10^4	2×10^5	5×10^5	-	3×10^8
N_0	2×10^3	-	-	10^9	10^4
T_c (μK)	0.17	0.4	2	50	0.015
n_0 (cm^{-3})	2.5×10^{12}	2×10^{12}	1.5×10^{14}	1.8×10^{14}	1×10^{12}
$n_0^{-1/3}$ ($\mathring{A}$)	$7,368$	$7,937$	$1,882$	$1,771$	10,000

BOSONS	${}^{4}_{2}He$	${}^{41}_{19}K$	${}^{133}_{55}Cs$	${}^{174}_{70}Yb^*$	${}^{52}_{24}Cr$
Year/Ref.	2001[20]	2001[21]	2003[22]	2003[23]	2005[24]
N	8×10^6	-	2×10^7	10^7	1.3×10^8
N_0	5×10^6	10^4	6.5×10^4	5×10^3	5×10^4
T_c (μK)	4.7	0.16	0.046	0.73	0.7
n_0 (cm^{-3})	3.8×10^{13}	6×10^{11}	1.3×10^{13}	7×10^{14}	-
$n_0^{-1/3}$ ($\mathring{A}$)	$2,974$	$11,856$ **max**	$4,253$	$1,126$ **min**	-

*of five stable isotopes.

Table 4. Ultra-cold fermionic-atom BEC (sometimes dubbed the "*VIth state of matter*") experimental parameters associated with trapped fermionic gases in which BEC has been observed to Dec. 2007, N and N_0 being the number of atoms in the initial cloud and in the condensate, respectively; T_c the BEC transition temperature, n_0 the reported boson (or peak atom) number density of the condensate at T_c in cm^{-3} and $n_0^{-1/3}$ is average interbosonic spacing in $\mathring{A}$. The lowest recorded temperature $T \simeq 45 \times 10^{-5} \mu K$ Ref. 28 is ~ 0.03 *lower* than lowest BEC critical temperature from Table 2 which is $T_c^{BEC}({}^{85}Rb) = 0.015 \mu K$.

FERMIONS	${}^{6}_{3}Li$	${}^{40}_{19}K$	${}^{173}_{70}Yb$ (of 2 stable isotopes)
Year/Ref.	2003 [25]	2003 [26]	2007 [27]
N	3.5×10^7	1.4×10^6	-
N_0	9×10^5	-	-
T_c (μK)	0.6	0.07	cooled to $T/T_F = 0.37$
n_0 (cm^{-3})	7×10^{13}	7×10^{12}	-
$n_0^{-1/3}$ ($\mathring{A}$)	$2,426$	$5,228$	-

5. Conclusion

We conclude that hole-Cooper-pairs play a significant role in determining the value of the critical generalized BEC temperature at *all* temperatures, at least in superconductors, just as antibosons do in the relativistic ideal Bose gas problem at higher temperatures where antibosons appear in substantial numbers. However, given that in cold-atom fermion systems densities are so low, we conjecture that their role will be significantly diminished.

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Appendix A. Stability of both 2e- and 2h-CP pure GBEC phases

The Helmholtz free energy per unit volume is by definition

$$F(T, L^d, \mu, n_0, m_0)/L^d \equiv \mu n - P(T, L^d, \mu, n_0, m_0) \tag{A.1}$$

where P is the system pressure. A necessary condition for a stable equilibrium thermodynamic state is a minimum in F with respect to n_0 and m_0, at fixed T, fixed number-density of electrons $n = N/L^d$ and fixed electron chemical potential $\mu = \mu(T, n, n_0)$. Here we show that both $[\partial^2(F/L^d)/\partial n_0^2]_n$ and $[\partial^2(F/L^d)/\partial m_0^2]_n$ are positive definite in the GBEC theory. The thermodynamic potential per unit d-dimensional volume is[6]

$$\begin{aligned}\Omega(T, L^d, \mu,\, n_0,\, m_0)/L^d \equiv -P(T, L^d, \mu,\, n_0, m_0) = \\ \int_0^\infty d\epsilon N(\epsilon)\,[\epsilon - \mu - E(\epsilon)] - 2\,k_B T \int_0^\infty d\epsilon\, N(\epsilon)\, \ln\{1 + \exp[-\beta\, E(\epsilon)]\} + \\ +[E_+(0) - 2\,\mu]\, n_0 + k_B T \int_{0+}^\infty d\varepsilon\, M(\varepsilon)\, \ln\{1 - \exp[-\beta\{E_+(0) + \varepsilon - 2\mu\}]\} + \\ +[2\,\mu - E_-(0)]\, m_0 + k_B T \int_{0+}^\infty d\varepsilon\, M(\varepsilon)\, \ln\{1 - \exp[-\beta\{2\mu - E_-(0) + \varepsilon\}]\}\end{aligned}$$

where $N(\epsilon)$ and $M(\varepsilon)$ are the fermionic and bosonic, respectively, density of states, while $E_+(0)$ and $E_-(0)$ are the phenomenological zero-center-of-mass-momentum 2e- and 2h-CPs, respectively. Here, the fermion spectrum $E(\epsilon)$ and fermion energy gap $\Delta(\epsilon)$ are related according to

$$E(\epsilon) = \sqrt{(\epsilon - \mu)^2 + \Delta^2(\epsilon)} \quad \textit{and} \quad \Delta(\epsilon) \equiv \sqrt{n_0} f_+(\epsilon) + \sqrt{m_0} f_-(\epsilon). \tag{A.2}$$

We first calculate $(\partial P/\partial n_0)_\mu$ and $(\partial P/\partial m_0)_\mu$, which are

$$(\partial P/\partial n_0)_\mu = -[E_+(0) - 2\,\mu] + \frac{f^2}{2} \int_{E_f}^{E_f + \hbar\omega_D} d\epsilon \frac{N(\epsilon)}{E(\epsilon)} \tanh \frac{1}{2} \beta E(\epsilon) \tag{A.3}$$

$$(\partial P/\partial m_0)_\mu = -[2\,\mu - E_-(0)] + \frac{f^2}{2} \int_{E_f - \hbar\omega_D}^{E_f} d\epsilon \frac{N(\epsilon)}{E(\epsilon)} \tanh \frac{1}{2} \beta E(\epsilon) \tag{A.4}$$

where from (A.2) $\partial E(\epsilon)/\partial n_0 = f^2/2E(\epsilon)$ was employed. Next, letting $x \equiv \beta E(\epsilon)/2$ one has

$$(\partial P^2/\partial n_0^2)_\mu = \frac{f^4 \beta^2}{16} \int_{E_f}^{E_f + \hbar\omega_D} d\epsilon \frac{N(\epsilon)}{E(\epsilon)} \frac{\partial}{\partial x} \left[\frac{\tanh x}{x} \right] < 0 \tag{A.5}$$

since $\partial(\tanh x/x)/\partial x$ can be shown[29] to be negative for all $x > 0$. Similarly, one finds that

$$(\partial P^2/\partial m_0^2)_\mu = \frac{f^4 \beta^2}{16} \int_{E_f - \hbar\omega_D}^{E_f} d\epsilon N(\epsilon) \frac{1}{E(\epsilon)} \frac{\partial}{\partial x} \left[\frac{\tanh x}{x} \right] < 0. \tag{A.6}$$

We then calculate

$$\left[\frac{\partial(F/L^d)}{\partial n_0}\right]_n = n\left[\frac{\partial \mu}{\partial n_0}\right]_n - \left[\frac{\partial P}{\partial n_0}\right]_n$$

$$= n\left[\frac{\partial \mu}{\partial n_0}\right]_n - \left[\frac{\partial P}{\partial n_0}\right]_\mu - \left[\frac{\partial P}{\partial \mu}\right]_{n_0}\left[\frac{\partial \mu}{\partial n_0}\right]_n = -\left[\frac{\partial P}{\partial n_0}\right]_\mu \tag{A.7}$$

since $n = [\partial P/\partial \mu]_{n_0}$. Next, for the pure 2e-CP GBEC phase one determines that

$$\left[\frac{\partial^2(F/L^d)}{\partial n_0^2}\right]_n = -\left[\frac{\partial}{\partial n_0}\right]_n\left[\frac{\partial P}{\partial n_0}\right]_\mu = -\left[\frac{\partial^2 P}{\partial n_0^2}\right]_\mu - \left[\frac{\partial \mu}{\partial n_0}\right]_n\left[\frac{\partial}{\partial \mu}\right]_{n_0}\left[\frac{\partial P}{\partial n_0}\right]_\mu$$

$$= -\left[\frac{\partial^2 P}{\partial n_0^2}\right]_\mu + \left[\frac{\partial \mu}{\partial n}\right]_{n_0}\left[\frac{\partial n}{\partial n_0}\right]_\mu\left[\frac{\partial^2 P}{\partial \mu \partial n_0}\right] = -\left[\frac{\partial^2 P}{\partial n_0^2}\right]_\mu + \left[\frac{\partial n}{\partial \mu}\right]_{n_0}^{-1}\left[\frac{\partial n}{\partial n_0}\right]_\mu^2 > 0$$

where (A.5) was employed. The derivation to here closely follows that of Ref. 30.

Finally, we turn to hole-CPs where one can similarly show for the pure 2h-CP GBEC phase that

$$\left[\frac{\partial^2(F/L^d)}{\partial m_0^2}\right]_n = -\left[\frac{\partial^2 P}{\partial m_0^2}\right]_\mu + \left[\frac{\partial n}{\partial \mu}\right]_{m_0}^{-1}\left[\frac{\partial n}{\partial m_0}\right]_\mu^2 > 0$$

where (A.6) was used. **QED**. Thus, both pure phases are stable, equilibrium thermodynamic states. However, the 2h phase exhibits an unacceptable divergent T_c as $n/n_f \to 0$ which will be investigated elsewhere.

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