International Journal of Modern Physics B
Vol. 23, Nos. 20 & 21 (2009) 4129–4137


World Scientific
www.worldscientific.com

# ON THE QUANTUM HALL EFFECT IN GRAPHENE

SHIGEJI FUJITA and JEONG-HYUK KIM
*Department of Physics, University at Buffalo, SUNY, Buffalo, NY 14260, USA*

KEI ITO
*The National Center for University Entrance Examinations, Meguro-ku, Tokyo 153-8501, Japan*

MANUEL DE LLANO
*Universidad Nacional de Mexico, AP 70-360, 04510 Mexico, D.F., Mexico*

The unusual quantum Hall effect (QHE) in graphene is often discussed in terms of Dirac fermions moving with a linear dispersion. A new theory describing the same phenomena is presented in terms of the more traditional composite bosons. The "electron" (wave packet) is shown to move easier in the direction [110] ≡ [110 $c$-axis] of the honeycomb lattice than perpendicular to it, while the "hole" moves easier in [001]. Since "electrons" and "holes" move in different channels, the number densities can be very high especially when the Fermi surface has "necks". The strong QHE at filling factor $\nu = 2$ arises from the "neck" Fermi surfaces.

*Keywords*: Graphene; quantum Hall effect; neck Fermi surface.

## 1. Introduction

Experiments[1,2,3] indicate that there are two kinds of oscillations for the magnetoresistivity $\rho$ in graphene when plotted as a function of the external magnetic field (magnitude) $B$. A Shubnikov-de Haas (SdH) oscillation appears on the low field side and a Quantum Hall Effect (QHE) oscillation appears on the high field side. We present a microscopic theory. We start with the graphene crystal, construct a two-dimensional Fermi surface, develop a Bardeen-Cooper-Schrieffer (BCS)-like theory[4] based on the phonon exchange attraction between the electron and the flux quantum (fluxon), and describe the phenomena.

The two-dimensional (2D) Landau Levels (LL) generates an oscillatory density of states. If multiple oscillations occur within the drop of the Fermi distribution function, then the SdH oscillation is developed for the magnetoconductivity. Thus, the carriers in the SdH must be fermions. The envelope of the SdH oscillation is controlled by the dressed electrons, see below. Its observed shrinkage toward the low fields[2] indicate that the dressed electron has a mass, see below.

An "electron" ("hole") is an elementary excitation which has an energy higher (lower) than the Fermi energy and which circulates counterclockwise (clockwise) viewed from the tip of the applied magnetic field vector. In the presence of a static magnetic field the classical electron orbit is curved. Then, the basic kinetic theoretical picture breaks down. Fortunately, quantum theory can save the situation. If the magnetic field is applied, the classical electron can continuously change from the straight line motion at zero field to the curved motion at a finite $B$. When the magnetic field is applied slowly, the energy of the electron does not change but the spiral motion always acts so as to reduce the magnetic fields. Hence, the total energy of the electron with its surrounding fields is less than the electron energy plus the unperturbed field energy. The electron "dressed" with the fields is in a bound (negative energy) state, and it is stable against the break-up. The guiding center of circulation can move in all directions in the absence of the electric field. If a weak electric field is applied, then the dressed electron whose position is the guiding center, preferentially moves in the field direction, and generates a current. We may apply kinetic theory to the guiding center motion. The dressed electron carries a magnetotransport mass $M^*$ different from the cyclotron mass $m^*$. The dressed electron can be identified as the composite particle[5] used in the thoery of quantum Hall effect[6]. Breifly, the electron circulates around a finite number of fluxons intact according to Onsager's flux quantization hypothesis[7]. Applying relativity, we may view that the fluxons circulate around the electron. From this view the electron is thought to carry a number of fluxons. The dressed electron is, then, a composite of an electron and fluxons. The composite particle moves as a fermion (boson) if it carries an even (odd) number of fluxons[8]. The composite boson with one fluxon is relevant for the QHE in graphene.

The integer QHE at filling factor $\nu = 1, 2, \ldots$ was found by Kim's group[3] at high magnetic fields ($\sim 45$ T), and was interpreted in terms of the Dirac fermion with a linear dispersion relation. However, the magnetotransport in general must be discussed in terms of the dressed electron, that is, the c-particles containing electrons and fluxons. The bare Dirac particle, if exist, would be dressed with fluxons in the presence of a magnetic field, and it would acquire a mass. As we see later, the conduction electron ("electron", "hole") has a size of the unit cell. It is unlikely that the point-like Dirac electron can contribute to a charge transport in solids. We shall develop an alternative microscopic theory.

## 2. Theory

In graphene carbon atoms (C) occupy the two-dimensional (2D) honeycomb crystal lattice.

The applied gate voltage can control the carrier charge type, "electron" (1) or "hole" (2), and the number density $n_j$, $j = 1, 2$. The "hole" has a positive charge $e$ and has a size of the unit hexagon formed by positively charged carbon ions $C^+$. The "hole" tends to stay away from $C^+$. The Center-of-Mass (CM) of the "hole" (wave

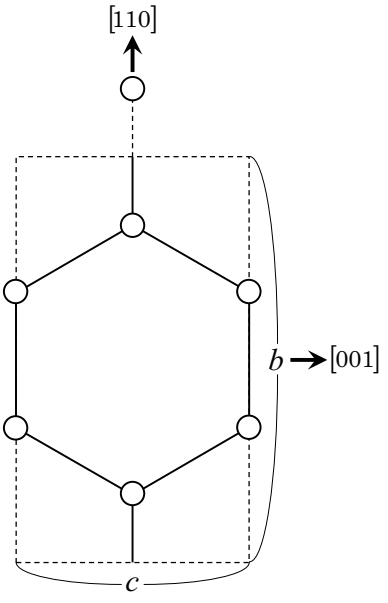


Fig. 1. A unit cell for graphene.

packet) is at the center of the hexagon. It moves easily with a small effective mass along the directions $\langle 100 \rangle \equiv \langle 100c\text{-axis} \rangle$, where we used the conventional Miller indices for the hexagonal lattice with the omission of the $c$-axis index. see below.

The "electron" has a negative charge $(-e)$ and a unit hexagon size. The "electron" tends to stay near $C^+$ and the charge distribution are concentrated near the hexagon. The CM of the "electron" (wave packet) is also at the center of the hexagon. It moves easily with a small effective mass along the directions $\langle 110 \rangle$, see Fig. 1 and below.

For the description of the electron motion in terms of the mass tensor it is convenient to introduce Cartesian coordinates, which may or may not align along the crystal's natural (triangular) axes. The "electron" may move up or down to the neighboring hexagon sites passing over one $C^+$. The positively charged $C^+$ acts as a welcome potential valley for the negatively charged "electron", while the $C^+$ acts as a hindering potential hill for the positively charged "hole". The "hole" however can move over on a series of vacant site, each surrounded by six C, without meating the hindering potential hills. Thus the easy channel direction for the "electron" and "hole" are $\langle 110 \rangle$ and $\langle 100 \rangle$, respectively.

We may choose the unit cell as shown. The choice is not unique. But the size of the rectangle with side-length pair $(b, c)$ for any unit cell is the same. Then, the Brillouin zone is unique: a rectangle with the side length $(2\pi/b, 2\pi/c)$. Let us consider the system (graphene) at 0 K. If we put an electron in the crystal, then the electron should occupy the center O of the Brillouin zone, where the lowest

energy lies. Additional electrons occupy the points neighboring O in consideration of Pauli's exclusion principle. The electron distribution is lattice-periodic over the entire crystal in accordance with the Bloch theorem.

The graphene is quadrivalent metal. The first few low-lying bands are completely filled. The uppermost partially filled bands are important for the transport properties discussion. We consider such a band. The Fermi surface, which defines the boundary between the filled and unfilled $k$-spaces (areas) is not a circle since the $x$-$y$ symmetry is broken. The effective mass is lighter in the direction [110] than perpendicular to it, and hence, the Fermi surface may more quickly grow in this direction with increasing number of "electrons". By the inversion symmetry of the crystal the Fermi surface must approach perpendicular to the Brillouin boundary. As the gate voltage is varied to the charge-neutral point, the Fermi surface should go through a "neck" configuration, where the density of states rapidly grows on both sides of the voltage, generating high densities of "electrons" and "holes". Experiments by Kim's group[2] indicate that (a) both "electrons" and "holes" can be excited in graphene, (b) At zero gate voltage the electrons are dominant, (c) the resistivity $\rho$ exhibits a sharp maximum at the "electron" density $n_1 \approx 2 \times 10^{11}$ cm$^2$. The feature (b) should arise from the existence of the welcoming $C^+$-potential for the "electrons". The feature (c) is due to the fact that the conductivity

$$\sigma \equiv \rho^{-1} = \frac{e^2 n_e}{m^*}\tau, \tag{1}$$

where $\tau$ is the relaxation time, must decrease since the effective mass $m^*$ shoot up to infinity in the small "neck" limit.

We note that the "neck" Fermi surface was observed in copper, which has high densities of "electrons" and "holes". Copper has negative Hall coefficient and positive thermopower. It is interesting to see if graphene shows the same properties.

The same easy channels in which the "electron" runs with a small mass, may be assumed for other hexagonal directions, [011] and [101]. Thus, the system does not show anisotropy.

We use a composite (c)-particle (fermion, boson) model[5] for the discussion of the QHE[6]. These particles, by definition, are bound. We assume a phonon-exchange attraction for the cause of the binding. Since the phonon (boson) does not carry a charge, the generation of the c-particles must occur without the charge-state change. It then follows that the phonon annihilation can, and must, pair-creates positively (+) and negatively (−) charged particles simultaneously. The numbers of + and − c-bosons created must be equal to each other. Normally, the carrier ("electron", "hole") densites are highly unbalanced. The "electrons" are the majority carrier for graphene at zero magnetic field, and the c-boson density is considerably smaller than the majority carrier density. In graphene however, the two necks (Fermi surface), which are assumed to be located near the filling factor $\nu = 2$, make the c-boson density very high.

The c-bosons move with linear dispersion relation[10]

$$W_p^{(j)} = W_0 + (2/\pi)v_F^{(j)}p, \tag{2}$$

where $W_0$ is the (negative) ground state energy, $p$ the c-boson momentum magnitude, and $v_F^{(j)}$ the Fermi velocity of type $j$, $j = 1$ ("electron"), $j = 2$ ("hole"). This relation is obtained, starting with a BCS-like Hamiltonian, setting up and solving an energy-eigenvalue problem for the moving c-boson. For completeness, we give a brief derivation in Appendix. These bosons move similar to massless particles with the common speed $(2/\pi)v_F^{(j)}$.

In graphene the "electron" c-bosons, having the greater speed, dominate the transport and the Bose-Einstein Condensation (BEC). The critical temperature $T_c$ below which the QHE is observed is given by[11]

$$k_B T_c = 1.24\,\hbar v_F^{(1)} n_0^{1/2}, \tag{3}$$

where $n_0$ is the c-boson density. Briefly the BEC occurs when the chemical potential $\mu$ vanishes at a finite $T$. The critical temperature $T_c$ can be determined from

$$n = (2\pi\hbar)^{-2}\int d^2p[e^{\beta_c\epsilon} - 1]^{-1}, \qquad \beta_c \equiv (k_B T_c)^{-1}. \tag{4}$$

After expanding the integrand in powers of $e^{-\beta_c\epsilon}$ and using $\epsilon = cp$, we obtain

$$n = 1.654(2\pi)^{-1}(k_B T_c/\hbar c)^2, \tag{5}$$

yielding formula (3) with $c = (2/\pi)v_F$.

## 3. Discussion

Novoselov *et al.*[12] observed a QHE in graphene at 300 K, which is most remarkable. Experiments[1] indicate that the Fermi velocity $v_F$ is approximately equal to 1.57 $\times 10^6$ ms$^{-1}$. Using Eq. (3), we obtain the critical temperature $T_c = 516$ K, for $n_0 = 10^{10}$ cm$^{-2}$. These numbers are reasonable.

The QHE behavior observed for graphene is remarkably similar to that for the heterojunction GaAs/AlGaAs. The physical conditions are different since the gate voltage and the applied magnetic field are varied. The present authors regard the QHE as the superconductivity induced by the magnetic field[10]. The magnetoresistivity for a QHE system reaches zero (supercurrent). The accompanied Hall resistivity has a plateau by kind of the Meissner effect. The QHE state is stable because of the energy gap in the c-boson excitation spectrum. If an extra magnetic field is applied to the system at optimum QHE state (the center of the plateau), then the sytem tries to stay in the same state by expelling the extra field. If the field is reduced, then the system stays in the same state by sucking in the extra field, thus generating a Hall resistivity plateau. In the graphene experiments, the gate voltage is varied. A little extra voltage relative to the voltage at the center of zero resistivity line charge the system without changing the superconducting state. The stationary charge in the form of "electrons" or "holes" do not change the supercurrent and the

Hall voltage, and hence the system remains the same state, keeping zero resistivity and the flat Hall resistivity. This state has an extra electric field energy

$$\frac{A}{2}\epsilon_0(\Delta E)^2, \tag{6}$$

where $A$ is the sample area, and $\Delta E$ is the electric field, positive or negative, generated by the sample charge. If the gate voltage is further increased (or decreased), then it will eventually destroy the superconducting state and the resistivity will rise from zero. This explains the QHE behavior.

Lanzara's group[13] used angle-resolved photoemission spectroscopy (ARPES) to study the dispersion relation. They found a linear dispersion relation of the form (2) in $\langle 110 \rangle$. Earlier Lanzara *et al.* obtained linear dispersion relations for the Cooper pairs in the HTSC[14]. We note that the HTSC and the QHE are similar. Both occur in 2D. The Cooper pairs and c-bosons move with linear dispersion relation of the form (2). The critical temperature $T_c$ is given by the same expression, Eq. (3).

**Acknowledgments**

The authors wish to thank Professors Eva Andrei and Murthy Ganapathy for enlightening discussions.

## Appendix A. Derivation of Equation (2)

We assume that the magnetic field **B** is applied perpendicular to the plane. The 2D LL's

$$E = (N_L + 1/2)\hbar\omega_0, \tag{A.1}$$
$$\omega_0 \equiv eB/m^*,$$

with the states $(N_L, k_y)$ have a great degeneracy. We regard the fluxon as a half-spin fermion with zero mass and zero charge. Our view is supported by the fact that the magnetic (electric) flux line cannot (can) terminate at a sink, and hence the associated fluxon (photon) is fermionic (bosonic). No half-spin fermion can disappear spontaneously because of angular momentum conservation. Fujita and Morabito showed that the CM of *any composite* moves as a fermion (boson), that is, the CM momentum occupation number is limited to 0 or 1 (unlimited) if the composite contains an odd (even) number of elementary fermions. Hence, the composite containing an electron and $Q$ fluxons moves as a boson (fermions) if $Q$ is odd (even). The c-particle (boson, fermion) in the prevalent theories is defined as the complex comprising an electron and Chern-Simons statistical field objects. These objects are neither bosonic nor fermionic. Hence, the statistics of the c-particle cannot be discussed. In our theory the fluxons are fermions, and the quantum statistics of the c-particles is well founded. The countability of the fluxons has been well established. The de Haas-van Alphen oscillations are routinely analyzed using Onsager's formula based on the flux quantization : $B = (h/e)n_\phi$, where $n_\phi$ is the fluxon density. The

countability and statistics of the fluxons are fundamental quantum particle properties. Hence, they cannot be derived from any Hamiltonian and must be postulated.

The longtitudinal phonon, acoustic or optical, proceeding in [100] can generate a density wave, which affects the electron (fluxon) motion by the lattice-ionic charge displacement (current), establishing the electron (fluxon)-phonon interaction. The phonon exchange between an electron and a fluxon generates a transition in the electron states with the effective interaction:

$$V_{\rm ef} \equiv |V_q V_q'| \frac{\hbar\omega_q}{(\epsilon_{|\mathbf{k}+\mathbf{q}|s} - \epsilon_{ks})^2 - \hbar^2\omega_q^2}, \tag{A.2}$$

where $V_q$ ($V_q'$) is the electron (fluxon)-phonon interaction strength; the Landau quantum number $N_L$ is omitted; the bold $\mathbf{k}$ denotes the 2D guiding center momentum and the italic $k$ the magnitude. The interaction is attractive when the electron states before and after the exchange have the same energy as in the degenerate LL so that $V_{\rm ef} = -|V_q V_q'|(\hbar\omega_q)^{-1}$.

Following BCS, we start with a Hamiltonian $H$ with the phonon variables eliminated:

$$\begin{aligned} H = &\sum_{\mathbf{k}}\sum_{s} \epsilon_k^{(1)} n_{\mathbf{k}s}^{(1)} + \sum_{\mathbf{k}}\sum_{s} \epsilon_k^{(2)} n_{\mathbf{k}s}^{(2)} \\ &+ \sum_{\mathbf{k}}\sum_{s} \epsilon_k^{(3)} n_{\mathbf{k}s}^{(3)} - v_1 \sum_{\mathbf{q}}{}' \sum_{\mathbf{k}}{}' \sum_{\mathbf{k}'}{}' \sum_{s} \Big[ B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(1)} \\ &+ B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger} + B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(1)} + B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger} \Big], \end{aligned} \tag{A.3}$$

where $n_{\mathbf{k}s}^{(j)}$ is the number operator for the "electron" (1) ["hole" (2), fluxon (3)] at momentum $\mathbf{k}$ and spin $s$ with the energy $\epsilon_{ks}^{(j)}$. We represent the "electron" ("hole") number $n_{\mathbf{k}s}^{(j)}$ by $c_{\mathbf{k}s}^{(j)\dagger} c_{\mathbf{k}s}^{(j)}$, where $c$ ($c^\dagger$) are annihilation (creation) operators satisfying the Fermi anticommutation rules:

$$\begin{aligned} &\{c_{\mathbf{k}s}^{(i)}, c_{\mathbf{k}'s'}^{(j)\dagger}\} \equiv c_{\mathbf{k}s}^{(i)} c_{\mathbf{k}'s'}^{(j)\dagger} + c_{\mathbf{k}'s'}^{(j)\dagger} c_{\mathbf{k}s}^{(i)} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}\delta_{i,j}, \\ &\{c_{\mathbf{k}s}^{(i)}, c_{\mathbf{k}'s'}^{(j)}\} = 0. \end{aligned} \tag{A.4}$$

We represent the fluxon number $n_{\mathbf{k}s}^{(3)}$ by $a_{\mathbf{k}s}^\dagger a_{\mathbf{k}s}$, with $a(a^\dagger)$, satisfying the anticommutation rules. $B_{\mathbf{k}\mathbf{q}\ s}^{(1)\dagger} \equiv c_{\mathbf{k}+\mathbf{q}/2\ s}^{(1)\dagger} a_{-\mathbf{k}+\mathbf{q}/2\ -s}^\dagger$, $B_{\mathbf{k}\mathbf{q}\ s}^{(2)} \equiv c_{\mathbf{k}+\mathbf{q}/2\ s}^{(2)} a_{-\mathbf{k}+\mathbf{q}/2\ -s}$. The prime on the summation means the restriction: $0 < \epsilon_{ks}^{(j)} < \hbar\omega_D$, $\omega_D =$ Debye frequency. If the fluxons are replaced by the conduction electrons ("electrons", "holes") our Hamiltonian $H$ is reduced to the original BCS Hamiltonian, Eq. (24) of Ref. 4. The "electron" and "hole" are generated, depending on the energy contour curvature sign. For example only "electrons" ("holes") are generated for a circular Fermi surface with the negative (positive) curvature whose inside (outside) is filled with electrons. Since the phonon has no charge, the phonon exchange cannot change the net charge. The pairing interaction terms in Eq. (A.3) conserve

the charge. The term $-v_0 B^{(1)\dagger}_{\mathbf{k'qs}} B^{(1)}_{\mathbf{kqs}}$, where $v_0 \equiv |V_q V'_q|(\hbar\omega_0 A)^{-1}$, $A =$ sample area, is the pairing strength, generates the transition in the "electron" states. Similarly, the exchange of a phonon generates a transition in the "hole" states, represented by $-v_0 B^{(2)}_{\mathbf{k'qs}} B^{(2)\dagger}_{\mathbf{kqs}}$. The phonon exchange can also pair-create and pair-annihilate "electron" ("hole")-fluxon composites, represented by $-v_0 B^{(1)\dagger}_{\mathbf{k'qs}} B^{(2)\dagger}_{\mathbf{kqs}}$, $-v_0 B^{(2)}_{\mathbf{k'qs}} B^{(1)}_{\mathbf{kqs}}$. At 0 K the system can have equal numbers of $-(+)$c-bosons, "electrons" ("holes") composites, generated by $-v_0 B^{(1)\dagger}_{\mathbf{k'qs}} B^{(2)\dagger}_{\mathbf{kqs}}$.

Let us take a dilute system of electrons moving in the plane. Applying a magnetic field $\mathbf{B}$ perpendicular to the plane, each electron will be in the LL state with the energy given by Eq. (A.1). In this state the electron can be viewed as circulating around the guiding center. We now apply a weak electric field $\mathbf{E}$ in the $x$-direction. With the scatterers (impurities, phonons) present in the system the guiding centers can jump from place to place preferrentially and generate a current in the $x$-direction.

The c-bosons, each with one electron and one fluxon, is relevant for graphene. Their energies $w_q^{(j)}$ are obtained from

$$w_q^{(j)}\Psi(\mathbf{k},\mathbf{q}) = \epsilon^{(j)}_{|\mathbf{k}+\mathbf{q}|}\Psi(\mathbf{k},\mathbf{q}) - (2\pi\hbar)^{-2} v_0 \int' d^2k' \Psi(\mathbf{k}',\mathbf{q}), \tag{A.5}$$

where $\Psi(\mathbf{k},\mathbf{q})$ is the reduced wavefunction for the c-boson; we assumed zero fluxon energy since the fluxon is meant to describe static magnetic field only. For small $q$, we obtain

$$w_q^{(j)} = w_0 + (2/\pi) v_F^{(j)} q, \quad w_0 = \frac{-\hbar\omega_D}{\exp(v_0 D_0)^{-1} - 1}, \tag{A.6}$$

where $v_F^{(j)} \equiv (2\epsilon_F/m_j)^{1/2}$ is the Fermi velocity and $D_0 \equiv D(\epsilon_F)$ the density of states per spin. Note that the energy $w_q^{(j)}$ depends *linearly* on the momentum $q$.

## References


1. K. S. Novoselov *et al.*, *Nature* **438**, 197 (2005).
2. Y. Zhang, Y. W. Tan, H. L. Stormer, and P. Kim, *Nature* **438**, 201 (2005).
3. Y. Zhang, *et al.*, *Phys. Rev. Lett.* **96**, 136806 (2006).
4. J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).
5. S. C. Zhang, T. H. Hansson, and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989).
6. Z. F. Ezawa, *Quantum Hall Effect* (World Scientific, Singapore, 2000); see also R. E. Prange and S. M. Girvin, eds. *Quantum Hall Effect* (Springer-Verlag, New York, 1990); B. I. Halperin, P. A. Lee, and H. Read, *Phys. Rev.* **B47**, 7312 (1993)
7. L. Onsager, *Phil. Mag.* **43**, 1006 (1952).
8. H. A. Bethe and R. Jackiw, *Intermediate Quantum Mechanics*, 2nd ed. (Benjamin, New York, 1968), p. 23; P. Ehrenfest and J. R. Oppenheimer, *Phys. Rev.* **37**, 311 (1931); S. Fujita and D. L. Morabito, *Mod. Phys. Lett.* **B12** 1061 (1998).
9. N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders, Philadelphia, 1976), pp. 291–293.
10. S. Fujita and Y. Okamura, *Phys. Rev.* **B69**, 155313 (2004).

11. S. Fujita and S. Godoy, *Quantum Statistical Theory of Superconductivity*, (Plenum, New York, 1996), pp. 184–185.
12. K. S. Novoselov *et al.*, *Science* **315**, 1379 (2007).
13. S. Y. Zhou *et al.*, *Nature Physics* **2**, 595 (2006).
14. A. Lanzara *et al.*, *Nature* **412**, 510 (2001).