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Unified description of collective modes in superconductors and semiconductors with an exciton condensed phase

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It is shown that the Bethe–Salpeter approach, the Bardeen, Cooper, and Schrieffer (BCS) based vertex method, and a generalized random-phase approximation (GRPA) to the many-electron problem in the presence of a condensed quantum phase yield the same theoretical excitation spectrum $\omega(Q)$ to collective modes. This spectrum reveals a secondary peak in optical absorption in semiconductors that can be understood as signaling the existence of an excitonic Bose–Einstein condensate (BEC). The analysis shows as well that there is an additional, non-trivial linearly-dispersive " moving" Cooper-pair solution for superconductors in both weak and strong coupling.

1 Introduction The proper description of the linear response of charged many-fermion systems under a weak electromagnetic field is of fundamental physical interest when a transition to a quantum condensed phase is possible. This is the case of Bose–Einstein condensation (BEC) of excitons in semiconductors or the creation of Cooper pairs in superconductors. In the first instance, there is interest in seeking compelling evidence of the existence or not of an exciton condensed phase through experimental optical probes (e.g., see Ref. [1] for possible signatures of BEC of excitons). A question naturally arising in this connection is whether the formation of the condensed phase leads to a new peak in the absorption spectrum, which could serve as a fingerprint for the existence of BEC in semiconductors.

In the theoretical description of collective-excitation spectra one finds that the positions of excitonic peaks in the presence of a condensed phase have been calculated by applying the (a) generalized random-phase approximation (GRPA) [2, 3], (b) vertex-equation method [4], based on Bardeen, Cooper, and Schrieffer (BCS) theory that includes the ladder diagram contribution to the vertex function, and (c) Bethe–Salpeter equation (BSE) approach [5, 6]. Due to

the long-range behavior of the Coulomb interaction it is difficult to obtain any analytical results. Numerical calculations done by Chu and Chang [4] show that close to the exciton resonance the absorption spectrum exhibits a sharp exciton peak followed by a secondary peak. The existence of this double-peak absorption spectrum indicates the presence of BE-condensed excitons. In contrast, numerical calculations based on the GRPA [2, 3] do not show a second peak in the optical response. In *both* methods [2–4], numerical calculations require dealing with giant matrices (3000×3000 in Refs. [2, 3]). Two different numerical methods, namely the stability matrix method [2, 3] and the usual singularity-removal method [4], have been used to invert these giant matrices so that it is unclear whether the secondary peaks in the optical response are artifacts produced by the numerical methods needed to solve the vertex equation, or that the vertex-equation method and the GRPA provide different collective-excitation spectra.

As we shall see, the GRPA, the BSE approach, and the vertex-equation method provide exactly the same position of the collective modes at zero momentum wavevector $Q = 0$. At the point $Q = 0$ there exists a so-called Goldstone (or

trivial) mode $\omega(\boldsymbol{Q}=0)=0$. This trivial solution means that the photon energy needed for direct transitions is equal to the exciton chemical potential. The chemical potential varies as the density of excitons increases. Strictly speaking, the observation of this density-dependent blueshift in the absorption spectrum does not imply the presence of a condensate as it is a result of the energies gained by the parallel-spin particles in avoiding each other, plus the weakening the Coulomb attraction between the excitonic electron and hole due to the exclusion principle. The existence of the secondary peak, i.e., of a non-trivial mode $\omega(\boldsymbol{Q}=0)=\omega_0$, could be related to the singularity in the joint density of states of the quasiparticles due to the formation of a gap [4] so that this peak could imply the formation of a condensed phase. If we ignore the wavevector dependence of the gap function, the energy separation between the secondary peak and the exciton peak is exactly twice the average value of the gap function. The presence (or absence) of the secondary peak should be verified experimentally, but the lack of incentives on the experimental side force one to test the excitonic BSE approach by applying it to the collective-mode dispersion in a one-dimensional (1D) superconductor where it is possible to obtain analytical results. Since the BE-condensed excitons and the superconductivity due to the formation of Cooper pairs are manifestations of the same effect, namely spontaneous symmetry-breaking, the phenomenon of the double-peak optical response in semiconductors should be similar to the existence of two modes (a Goldstone or trivial mode and a *non-trivial* mode) in superconductors. If the numerical calculations by Chu and Chang are correct, then one can expect that the separation between the Goldstone and the non-trivial mode at $\boldsymbol{Q}=0$ should be twice the value of the gap.

Collective excitations in superconductors at $T=0$ were studied decades ago within the GRPA [7–9]. It was later pointed out that the GRPA is able to describe the evolution of the system from a large-Cooper-pair (CP) regime to a small, well-separated-CP regime in the attractive Hubbard model [10, 11] and in the attractive δ interaction Fermi gas [12]. Unfortunately, the determinants used to obtain the collective-mode dispersion in Refs. [11, 12] are only 3×3 while the correct determinant obtained by linearizing the Anderson–Rickayzen (AR) equations must be 4×4. The GRPA equations in Ref. [10] are equivalent to our BSE (1) below, where the screened Coulomb interaction $V_\mathrm{s}(\boldsymbol{k})$ is replaced by the strength U of the attractive interaction in the Hubbard model, but the expressions for $\gamma_{k,k',q}$ and $\delta_{k,k',q}$ (see Eq. 9a and b in Ref. [10]) are incorrect and thus cast doubts on all numerical calculations for the collective-mode energies in Ref. [10]. The correct expressions for $\gamma_{k,k',q}$ and $\delta_{k,k',q}$ can be obtained from (1) below. More recently, the spectrum of the collective excitations in superconductors has been examined by applying the BCS mean-field theory combined with the BSE for the two-particle bound states [13, 14]. The BSE approach used in Ref. [13] does not represent correctly the physics involved because it does not take into account the BSE amplitudes for electron–hole Ψ_eh, hole–electron Ψ_he,

and hole–hole Ψ_hh CPs. These amplitudes have been taken into account in Ref. [14], but the diagrammatic perturbation theory used to analyze the form of the BSE kernel provided incorrect expressions for the BSE so that conclusions about the collective-excitation spectrum are in doubt.

Here we use the BSE approach to calculate the collective-excitation spectra in superconductors assuming that the attractive interaction responsible for the formation of CPs is constant in momentum space. The main results are that: (i) the BSE and the GRPA give the same equations for collective modes; (ii) the energy separation between the Goldstone and the non-trivial mode at $\boldsymbol{Q}=0$ is exactly twice the superconducting gap; (iii) contrary to previous calculations [11, 12] our approach provides an equation for the collective-mode dispersion that depends on all four coherence factors and is given by setting the associated 4×4 determinant to 0, while in previous calculations [11, 12] collective-mode spectra does not depend on the fourth coherence factor $u_{k+Q}v_k - u_k v_{k+Q}$ so that the corresponding determinant is only 3×3.

2 Collective excitations in semiconductors
2.1 Bethe–Salpeter approach
We consider a direct-gap semiconductor with non-degenerate and isotropic bands. For simplicity, we assume equal electron and hole masses, i.e., $m_\mathrm{e}=m_\mathrm{h}\equiv m$. In this case the electron- and hole-dispersion relations relative to the corresponding chemical potential $\mu_{\mathrm{e,h}}=(E_\mathrm{g}\pm\mu)/2$ are ($\hbar=1$): $\varepsilon_\mathrm{e}(\boldsymbol{k})=\boldsymbol{k}^2/2m+E_\mathrm{g}-\mu_\mathrm{e}$ and $\varepsilon_\mathrm{h}(\boldsymbol{k})=-(\boldsymbol{k}^2/2m+\mu_\mathrm{h})$, respectively. Here $\mu=\mu_\mathrm{e}-\mu_\mathrm{h}$ is the exciton chemical potential, E_g is the semiconductor gap. The chemical potential μ is determined self-consistently as a non-trivial function of the electron number density n. In the presence of a BE-condensate, one can write the BSE for the exciton energy $\omega_\nu(\boldsymbol{Q})$ and corresponding wave functions $G_\nu^\pm(\boldsymbol{q},\boldsymbol{Q})$ (for details, see, e.g., Ref. [6]):

$$[\omega_\nu(\boldsymbol{Q})-\varepsilon(\boldsymbol{k},\boldsymbol{Q})]G_\nu^+(\boldsymbol{k},\boldsymbol{Q})$$
$$=-\frac{1}{2}\sum_q V_\mathrm{s}(\boldsymbol{k}-\boldsymbol{q})\big[\gamma_{k,Q}\gamma_{q,Q}+l_{k,Q}l_{q,Q}$$
$$+\widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q}+m_{k,Q}m_{q,Q}\big]G_\nu^+(\boldsymbol{q},\boldsymbol{Q})$$
$$-\frac{1}{2}\sum_q V_\mathrm{s}(\boldsymbol{k}-\boldsymbol{q})\big[\gamma_{k,Q}\gamma_{q,Q}-l_{k,Q}l_{q,Q}$$
$$+\widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q}-m_{k,Q}m_{q,Q}\big]G_\nu^-(\boldsymbol{q},\boldsymbol{Q}),$$

$$[\omega_\nu(\boldsymbol{Q})+\varepsilon(\boldsymbol{k},\boldsymbol{Q})]G_\nu^-(\boldsymbol{k},\boldsymbol{Q})$$
$$=\frac{1}{2}\sum_q V_\mathrm{s}(\boldsymbol{k}-\boldsymbol{q})\big[\gamma_{k,Q}\gamma_{q,Q}+l_{k,Q}l_{q,Q}$$
$$+\widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q}+m_{k,Q}m_{q,Q}\big]G_\nu^-(\boldsymbol{q},\boldsymbol{Q})$$
$$+\frac{1}{2}\sum_q V_\mathrm{s}(\boldsymbol{k}-\boldsymbol{q})\big[\gamma_{k,Q}\gamma_{q,Q}-l_{k,Q}l_{q,Q}$$
$$+\widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q}-m_{k,Q}m_{q,Q}\big]G_\nu^+(\boldsymbol{q},\boldsymbol{Q}),\qquad(1)$$

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where $\varepsilon(\boldsymbol{k}, \boldsymbol{Q}) \equiv E(\boldsymbol{k} + \boldsymbol{Q}) + E(\boldsymbol{k})$. The coherent factors are $\gamma_{k,Q} \equiv u_k u_{k+Q} + v_k v_{k+Q}, l_{k,Q} \equiv u_k u_{k+Q} - v_k v_{k+Q}, \widetilde{\gamma}_{k,Q} \equiv u_k v_{k+Q} - u_{k+Q} v_k$, and $m_{k,Q} \equiv u_k v_{k+Q} + u_{k+Q} v_k$. Here, u_p and v_p are the usual BCS coherence factors:

$$u_p^2 = \frac{1}{2}\left[1 + \frac{\varepsilon(\boldsymbol{p})}{E(\boldsymbol{p})}\right], \quad v_p^2 = \frac{1}{2}\left[1 - \frac{\varepsilon(\boldsymbol{p})}{E(\boldsymbol{p})}\right],$$

$$E(\boldsymbol{p}) = \sqrt{\varepsilon^2(\boldsymbol{p}) + \Delta^2(\boldsymbol{p})}. \tag{2}$$

The Goldstone mode solution of the above BSE corresponds to $G^+(\boldsymbol{q}, 0) = G^-(\boldsymbol{q}, 0)$. The presence of the excitonic BEC would be revealed by the non-trivial solution $\omega(\boldsymbol{Q} = 0) \equiv \omega_0 > 0$ of Eq. (1). This solution manifests itself as a secondary peak in the absorption spectrum.

2.2 Vertex-equation method This method leads to an expression for the absorption coefficient which depends linearly on a function $G(\boldsymbol{k})$ which in turn is a solution of the following inhomogeneous equation [4]:

$$\omega^2 G_k - \sum_q H_{k,q} G_q$$
$$= \sum_q [2E(\boldsymbol{k})\delta_{k,q} - V_s(\boldsymbol{k} - \boldsymbol{q})]F_q^0 - \omega \Gamma_2^0(\boldsymbol{k}), \tag{3}$$

where ω is the energy difference between the photon energy and the chemical potential, $F_q^0 \equiv l_{q,Q=0}\Gamma_1^0(\boldsymbol{q})$, and $\Gamma_i^0(\boldsymbol{q})$ (where $i = 1, \ldots, 4$) denotes the bare vertex functions. The Hamiltonian in (3) is:

$$H_{k,q} = \sum_p [2E(\boldsymbol{k})\delta(\boldsymbol{k} - \boldsymbol{p}) - V_s(\boldsymbol{k} - \boldsymbol{p})][2E(\boldsymbol{q})\delta(\boldsymbol{p} - \boldsymbol{q})$$
$$- V_s(\boldsymbol{p} - \boldsymbol{q})(l_p l_q + m_p m_q)]. \tag{4}$$

The solution of (3) may be obtained by expressing the left-hand side of (3) in its eigenspectrum representation:

$$G_k = \sum_\nu \sum_q \frac{S_{k,\nu} S_{q,\nu}^*}{\omega^2 - \omega_\nu^2}$$
$$\times \left(\sum_p [2E(\boldsymbol{q})\delta(\boldsymbol{q} - \boldsymbol{p}) - V_s(\boldsymbol{q} - \boldsymbol{p})]F_p^0 - \omega \Gamma_2^0(\boldsymbol{q})\right). \tag{5}$$

Here the ω_ν are the eigenvalues of $H_{k,q}$, and $S_{k,\nu}$ are the eigenvectors of:

$$\omega_\nu^2 S_{k,\nu} - \sum_q H_{kq} S_{q,\nu} = 0. \tag{6}$$

Obviously, the absorption coefficient will exhibit peaks at the same energies as those obtained by the BSE approach because Eq. (6) for $S_{k,\nu}$ follows from the BSE (1) with $S_{k,\nu} \equiv G_\nu^+(\boldsymbol{k}, \boldsymbol{Q})\big|_{Q=0} - G_\nu^-(\boldsymbol{k}, \boldsymbol{Q})\big|_{Q=0}$.

2.3 General random phase approximation The semiconductor AR equations [2, 3] provide the following BSE for the retarded Green function $\widehat{G}^Q(\boldsymbol{q}, \boldsymbol{p}; \omega)$:

$$[\omega \tau_0 + \varepsilon(\boldsymbol{k}, \boldsymbol{Q})\tau_3]\widehat{G}^Q(\boldsymbol{k}, \boldsymbol{p}; \omega)$$
$$- \sum_q V_s(\boldsymbol{k} - \boldsymbol{q})\Big[C_{k,q}^{(0)} C_{k+Q,q+Q}^{(0)} \tau_2$$
$$+ C_{k,q}^{(3)} C_{k+Q,q+Q}^{(3)} i\tau_1\Big]\widehat{G}^Q(\boldsymbol{q}, \boldsymbol{p}; \omega) = \delta_{k,p}(\tau_0 + \tau_2). \tag{7}$$

Here τ_i $(i = 1, 2, 3)$ are the Pauli spin matrices, and τ_0 is the 2×2 unit matrix. In Eq. (7) we have neglected the collective fluctuation term that appears, for example, in Eq. (27) of Ref. [3] since it will not modify the eigenvalues ω_ν (see below). The coherence factors are:

$$C_{k,q}^{(0)} \equiv u_k u_q + v_k v_q,$$
$$C_{k,q}^{(0)} C_{k+Q,q+Q}^{(0)} \equiv \gamma_{k,Q}\gamma_{q,Q} + l_{k,Q}l_{q,Q} + \widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q} + m_{k,Q}m_{q,Q},$$
$$C_{k,q}^{(3)} \equiv u_k v_q - v_k u_q,$$
$$C_{k,q}^{(3)} C_{k+Q,q+Q}^{(3)} \equiv \gamma_{k,Q}\gamma_{q,Q} - l_{k,Q}l_{q,Q} + \widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q} - m_{k,Q}m_{q,Q}. \tag{8}$$

The retarded Green function is then obtained by inverting the matrix:

$$\widehat{H}_{k,q}^Q = [\omega \tau_0 + \varepsilon(\boldsymbol{k}, \boldsymbol{Q})\tau_3]\delta_{k,q} - V_s(\boldsymbol{k} - \boldsymbol{q})$$
$$\times \Big[C_{k,q}^{(0)} C_{k+Q,q+Q}^{(0)} \tau_2 + C_{k,q}^{(3)} C_{k+Q,q+Q}^{(3)} i\tau_1\Big]. \tag{9}$$

Writing the inverse of matrix $\widehat{H}_{k,q}^Q$ as $\sum_\nu \widehat{S}_{k,\nu}^Q \widehat{S}_{q,\nu}^{Q\dagger}(\omega - \omega_\nu(\boldsymbol{Q}))^{-1}$, we find:

$$\widehat{G}^Q(\boldsymbol{k}, \boldsymbol{q}; \omega) = \sum_\nu \frac{\widehat{S}_{k,\nu}^Q \widehat{S}_{q,\nu}^{Q\dagger}}{\omega - \omega_\nu(\boldsymbol{Q}) + i0^+}(\tau_0 + \tau_2). \tag{10}$$

Here $\omega_\nu(\boldsymbol{Q})$ are the eigenvalues of $\widehat{H}_{k,q}^Q$ while the elements of the two-component vector $\widehat{S}_{k,\nu}^Q = \begin{pmatrix} \varphi_{k,\nu}^Q \\ \phi_{k,\nu}^Q \end{pmatrix}$ are its eigenfunctions. From Eq. (7) it is clear that the second term is:

$$C_{k,q}^{(0)} C_{k+Q,q+Q}^{(0)} \tau_2 + C_{k,q}^{(3)} C_{k+Q,q+Q}^{(3)} i\tau_1 =$$
$$\begin{pmatrix} 0 & -2i[l_{k,Q}l_{q,Q} + m_{k,Q}m_{q,Q}] \\ 2i[\gamma_{k,Q}\gamma_{q,Q} + \widetilde{\gamma}_{k,Q}\widetilde{\gamma}_{q,Q}] & 0 \end{pmatrix}. \tag{11}$$

Upon substitution in Eq (7), the components of $\widehat{S}_{k,\nu}^Q$ satisfy the following eigenproblem:

$$
\begin{aligned}
&\left[\omega_\nu^2(\boldsymbol{Q}) - \varepsilon^2(\boldsymbol{k},\boldsymbol{Q})\right]\phi_{k,\nu}^{\boldsymbol{Q}} \\
&= -\sum_q V_{\mathrm{s}}(\boldsymbol{k}-\boldsymbol{q})\varepsilon(\boldsymbol{k},\boldsymbol{Q})\left[\gamma_{k,\boldsymbol{Q}}\gamma_{q,\boldsymbol{Q}} + \widetilde{\gamma}_{k,\boldsymbol{Q}}\widetilde{\gamma}_{q,\boldsymbol{Q}}\right]\phi_{q,\nu}^{\boldsymbol{Q}} \\
&\quad - \sum_q V_{\mathrm{s}}(\boldsymbol{k}-\boldsymbol{q})\omega_\nu(\boldsymbol{Q})\left[l_{k,\boldsymbol{Q}}l_{q,\boldsymbol{Q}} + m_{k,\boldsymbol{Q}}m_{q,\boldsymbol{Q}}\right]\varphi_{q,\nu}^{\boldsymbol{Q}},
\end{aligned}
$$

$$
\begin{aligned}
&\left[\omega_\nu^2(\boldsymbol{Q}) - \varepsilon^2(\boldsymbol{k},\boldsymbol{Q})\right]\varphi_{k,\nu}^{\boldsymbol{Q}} \\
&= -\sum_q V_{\mathrm{s}}(\boldsymbol{k}-\boldsymbol{q})\varepsilon(\boldsymbol{k},\boldsymbol{Q})\left[l_{k,\boldsymbol{Q}}l_{q,\boldsymbol{Q}} + m_{k,\boldsymbol{Q}}m_{q,\boldsymbol{Q}}\right]\varphi_{q,\nu}^{\boldsymbol{Q}} \\
&\quad - \sum_q V_{\mathrm{s}}(\boldsymbol{k}-\boldsymbol{q})\omega_\nu(\boldsymbol{Q})\left[\gamma_{k,\boldsymbol{Q}}\gamma_{q,\boldsymbol{Q}} + \widetilde{\gamma}_{k,\boldsymbol{Q}}\widetilde{\gamma}_{q,\boldsymbol{Q}}\right]\phi_{q,\nu}^{\boldsymbol{Q}}.
\end{aligned}
\tag{12}
$$

It is easy to check that both the BSE approach and the GRPA provide the same collective-mode energies $\omega_\nu(\boldsymbol{Q})$. To prove this we introduced the two functions $Z_\nu^\pm(\boldsymbol{q},\boldsymbol{Q}) \equiv G_\nu^+(\boldsymbol{q},\boldsymbol{Q}) \pm G_\nu^-(\boldsymbol{q},\boldsymbol{Q})$. Multiplying first of Eq. (1) by $[\omega_\nu + \varepsilon(\boldsymbol{k},\boldsymbol{Q})]$ and the second one by $[\omega_\nu - \varepsilon(\boldsymbol{k},\boldsymbol{Q})]$ and adding them we again obtain Eq. (12) with $\varphi_{k,\nu}^{\boldsymbol{Q}} \equiv Z_\nu^-(\boldsymbol{q},\boldsymbol{Q})$ and $\phi_{k,\nu}^{\boldsymbol{Q}} \equiv Z_\nu^+(\boldsymbol{q},\boldsymbol{Q})$. Thus, we conclude that the BSE approach, the vertex equation method and the GRPA provide the *same* collective excitations $\omega_\nu(\boldsymbol{Q})$. The collective excitation spectrum at point $\boldsymbol{Q} = 0$ is the same, but it is still unknown, in principle, whether the secondary peaks in the paper by Chu and Chang [4] are due to their singularity-removal method. Thus, we have to check the BSE approach using a simple interaction where exact results can be obtained. A possible choice is a superconductor with a contact interaction.

3 Collective excitations in superconductors We start from the excitonic GRPA Eq. (12). The model allowing us to obtain analytical results assumes an interfermion interaction in a 1D superconductor that is an attractive contact interaction, i.e., its Fourier transform is a negative constant $-V_0$ for all momenta. This in turn leaves a wavevector independent gap function, i.e., $\Delta(\boldsymbol{k}) \equiv \Delta_0$. This leads to the number and gap

equations:

$$
n = \int \frac{d^d\boldsymbol{q}}{(2\pi)^d}\left[1 - \frac{\varepsilon(\boldsymbol{q})}{E(\boldsymbol{q})}\right], \quad 1 = \frac{1}{2}V_0 \int \frac{d^d\boldsymbol{q}}{(2\pi)^d}\frac{1}{E(\boldsymbol{q})}, \tag{13}
$$

where $\varepsilon(\boldsymbol{k}) \equiv k^2/2m - \mu$ and $E(\boldsymbol{k}) \equiv \sqrt{\varepsilon^2(\boldsymbol{k}) + \Delta_0^2}$. Starting from Eq. (12) where $V_{\mathrm{s}}(\boldsymbol{k}-\boldsymbol{q}) = -V_0$, and after some straightforward algebra we arrive at a set of four coupled homogeneous equations for $\Gamma(\boldsymbol{Q}), \Lambda(\boldsymbol{Q}), M(\boldsymbol{Q})$, and $\widetilde{\Gamma}(\boldsymbol{Q})$:

$$
\begin{pmatrix}
1 + V_0 I(\varepsilon;\gamma,\gamma) & V_0 I(\omega;\gamma,l) & 2V_0 I(\omega;\gamma,m) & V_0 I(\varepsilon;\gamma,\widetilde{\gamma}) \\
V_0 I(\omega;\gamma,l) & 1 + V_0 I(\varepsilon;l,l) & 2V_0 I(\varepsilon;l,m) & V_0 I(\omega;l,\widetilde{\gamma}) \\
\frac{1}{2}V_0 I(\omega;\gamma,m) & \frac{1}{2}V_0 I(\varepsilon;l,m) & 1 + V_0 I(\varepsilon;m,m) & \frac{1}{2}V_0 I(\omega;m,\widetilde{\gamma}) \\
V_0 I(\varepsilon;\gamma,\widetilde{\gamma}) & V_0 I(\omega;l,\widetilde{\gamma}) & 2V_0 I(\omega;m,\widetilde{\gamma}) & 1 + V_0 I(\varepsilon;\widetilde{\gamma},\widetilde{\gamma})
\end{pmatrix}
\begin{pmatrix}
\Gamma \\
\Lambda \\
M \\
\widetilde{\Gamma}
\end{pmatrix} = 0.
\tag{14}
$$

Here, we have introduced the following notations:

$$
\Gamma(\boldsymbol{Q}) = -\int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\gamma_{k,\boldsymbol{Q}}[G^+(\boldsymbol{k},\boldsymbol{Q}) - G^-(\boldsymbol{k},\boldsymbol{Q})], \tag{15}
$$

$$
\Lambda(\boldsymbol{Q}) = -\int \frac{d^d\boldsymbol{k}}{(2\pi)^d}l_{k,\boldsymbol{Q}}[G^+(\boldsymbol{k},\boldsymbol{Q}) + G^-(\boldsymbol{k},\boldsymbol{Q})], \tag{16}
$$

$$
M(\boldsymbol{Q}) = -\frac{1}{2}\int \frac{d^d\boldsymbol{k}}{(2\pi)^d}m_{k,\boldsymbol{Q}}[G^+(\boldsymbol{k},\boldsymbol{Q}) + G^-(\boldsymbol{k},\boldsymbol{Q})], \tag{17}
$$

$$
\widetilde{\Gamma}(\boldsymbol{Q}) = -\int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\widetilde{\gamma}_{k,\boldsymbol{Q}}[G^+(\boldsymbol{k},\boldsymbol{Q}) - G^-(\boldsymbol{k},\boldsymbol{Q})]. \tag{18}
$$

The quantities $I(\varepsilon;b,c)$ and $I(\omega;b,c)$ in the notation of Ref. [11] are defined as:

$$
I(\varepsilon;a,b) = \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\frac{\varepsilon(\boldsymbol{k},\boldsymbol{Q})a(\boldsymbol{k},\boldsymbol{Q})b(\boldsymbol{k},\boldsymbol{Q})}{\omega^2(\boldsymbol{Q}) - \varepsilon^2(\boldsymbol{k},\boldsymbol{Q})},
$$

$$
I(\omega;a,b) = \int \frac{d^d\boldsymbol{k}}{(2\pi)^d}\frac{\omega(\boldsymbol{Q})a(\boldsymbol{k},\boldsymbol{Q})b(\boldsymbol{k},\boldsymbol{Q})}{\omega^2(\boldsymbol{Q}) - \varepsilon^2(\boldsymbol{k},\boldsymbol{Q})},
$$

with the factors $\{a(\boldsymbol{k},\boldsymbol{Q}),b(\boldsymbol{k},\boldsymbol{Q})\}$ denoting any of the four coherence factors $\{l_{k,\boldsymbol{Q}},m_{k,\boldsymbol{Q}},\gamma_{k,\boldsymbol{Q}},\widetilde{\gamma}_{k,\boldsymbol{Q}}\}$. The spectrum of the collective excitations $\omega(\boldsymbol{Q})$ could be obtained assuming the vanishing of the determinant:

$$
\begin{vmatrix}
1 + V_0 I(\varepsilon;\gamma,\gamma) & V_0 I(\omega;\gamma,l) & 2V_0 I(\omega;\gamma,m) & V_0 I(\varepsilon;\gamma,\widetilde{\gamma}) \\
V_0 I(\omega;\gamma,l) & 1 + V_0 I(\varepsilon;l,l) & 2V_0 I(\varepsilon;l,m) & V_0 I(\omega;l,\widetilde{\gamma}) \\
\frac{1}{2}V_0 I(\omega;\gamma,m) & \frac{1}{2}V_0 I(\varepsilon;l,m) & 1 + V_0 I(\varepsilon;m,m) & \frac{1}{2}V_0 I(\omega;m,\widetilde{\gamma}) \\
V_0 I(\varepsilon;\gamma,\widetilde{\gamma}) & V_0 I(\omega;l,\widetilde{\gamma}) & 2V_0 I(\omega;m,\widetilde{\gamma}) & 1 + V_0 I(\varepsilon;\widetilde{\gamma},\widetilde{\gamma})
\end{vmatrix} = 0.
\tag{19}
$$

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The 3×3 determinant of Refs. [11, 12] coincides with the corresponding cofactor of the element in the fourth row and the fourth column of our determinant (19). Only at $Q = 0$ does the 3×3 determinant coincide with the determinant (19) since $\widetilde{\gamma} = 0$ when $Q = 0$ and $I(\varepsilon; \widetilde{\gamma}, \widetilde{\gamma}) \to I(\varepsilon; 0, 0) = 0$.

3.1 Goldstone mode in the weak-coupling limit
In what follows we use a 1D superconductor to calculate the dispersion of the collective modes using Eq. (19). In the weak coupling limit the chemical potential tends to the Fermi energy $\varepsilon_F = k_F^2/2m$, and therefore, the integrals are peaked at ε_F. The gap equation assumes the form:

$$1 = \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{(k^2 - 1)^2 + \Delta^2}}$$

$$= \frac{\lambda}{2} \int dE \frac{N(E)}{\sqrt{E^2 + \Delta^2}}. \tag{20}$$

Here the integration is over the dimensionless variables k and E, and $N(E) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \delta(E - (k^2 - 1))$ is the dimensionless density of states. In what follows, we introduce the following dimensionless variables: $\lambda = V_0 k_F/(2\pi\varepsilon_F)$, $q = Q/k_F$, and $\Delta = \Delta_0/\varepsilon_F$. Since the above integrals are peaked about the Fermi level, we assume that values of the integrals may be estimated by: (i) keeping only a thin shell of thickness $2\Omega_D$, where Ω_D is the Debye energy, and (ii) assuming a constant density of states $N(0)$ inside this thin shell. In this approximation the gap equation is replaced by:

$$1 = \frac{\lambda N(0)}{2} \int_{-\Omega_D/\varepsilon_F}^{\Omega_D/\varepsilon_F} dE \frac{1}{\sqrt{E^2 + \Delta^2}}$$

$$= \lambda N(0) \ln \left(\frac{\Omega_D + \sqrt{\Omega_D^2 + \Delta_0^2}}{\Delta_0} \right). \tag{21}$$

In week-coupling limit $(\Omega_D \gg \Delta_0)$ the gap is:

$$\Delta_0 = 2\Omega_D \exp \left(-\frac{1}{\lambda N(0)} \right). \tag{22}$$

We are looking for the Goldstone collective mode with a dispersion $\Omega(Q) = c\hbar Q$, where c is the speed of sound. In the weak-coupling case we use a small-Q, small-Δ_0, and small-ω expansions of the terms in the determinant (19). Solving for $\omega = \Omega(Q)/\varepsilon_F$ we obtain the following speed of sound: $c = [1 - \lambda N(0)]^{1/2} \hbar k_F/m$. Thus, the BSE approach confirms the result obtained by the GRPA that the Goldstone mode in a weak-coupling is a phonon-like excitation, and its speed of sound does not depend on the gap [11].

3.2 Non-trivial mode in the weak-coupling limit
The coherence factors $l_{k,Q}$ and $\widetilde{\gamma}_{k,Q}$ are odd under change of the sign of $\varepsilon(\boldsymbol{k})$, while the other two coherence

factors are even. This means that $I(\omega; \gamma, l)$, $I(\varepsilon; l, m)$, $I(\varepsilon; \gamma, \widetilde{\gamma})$, $I(\omega; m, \widetilde{\gamma})$ are vanishing integrals, and (19) assumes the form:

$$([1 + V_0 I(\varepsilon; l, l)][1 + V_0 I(\varepsilon; \widetilde{\gamma}, \widetilde{\gamma})] - [V_0 I(\omega; l, \widetilde{\gamma})]^2)$$

$$\times \begin{vmatrix} 1 + V_0 I(\varepsilon; \gamma, \gamma) & 2V_0 I(\omega; \gamma, m) \\ \dfrac{1}{2} V_0 I(\omega, m, \gamma) & 1 + V_0 I(\varepsilon; m, m) \end{vmatrix} = 0. \tag{23}$$

The non-trivial collective mode has a dispersion $\omega(Q) = \omega_0 + v\hbar Q$, where v is the speed of sound and $\omega_0 > 0$. In week-coupling limit we repeat the small-Q and small-Δ_0 expansions of the terms in the above 2×2 determinant. First, we shall check about the existence of a non-trivial solution with $0 < \omega_0 < 2\Delta_0$ at point $Q = 0$. In this interval we have:

$$1 + V_0 I(\varepsilon; \gamma, \gamma)$$

$$\to -\frac{\omega_0 \lambda N(0) \arctan \left(\dfrac{\omega_0}{\sqrt{4\Delta_0^2 - \omega_0^2}} \right)}{\sqrt{4\Delta_0^2 - \omega_0^2}}, \tag{24}$$

$$V_0 I(\omega; \gamma, m)$$

$$\to -\frac{2\Delta_0 \lambda N(0) \arctan \left(\dfrac{\omega_0}{\sqrt{4\Delta_0^2 - \omega_0^2}} \right)}{\sqrt{4\Delta_0^2 - \omega_0^2}}, \tag{25}$$

$$1 + V_0 I(\varepsilon; m, m)$$

$$\to 1 - \frac{4\Delta_0^2 \lambda N(0) \arctan \left(\dfrac{\omega_0}{\sqrt{4\Delta_0^2 - \omega_0^2}} \right)}{\omega_0 \sqrt{4\Delta_0^2 - \omega_0^2}}, \tag{26}$$

$$1 + V_0 I(\varepsilon; l, l)$$

$$\to 1 - \frac{\sqrt{4\Delta_0^2 - \omega_0^2} \lambda N(0) \arctan \left(\dfrac{\omega_0}{\sqrt{4\Delta_0^2 - \omega_0^2}} \right)}{\omega_0}. \tag{27}$$

Thus, in the interval $0 < \omega_0 < 2\Delta_0$, there are no solutions of Eq. (23). The only possible solution for $\{\omega_0, \Delta_0\} \ll \Omega_D$ is $\omega_0 = 2\Delta_0$. In this case the 2×2 determinant in (23) is not 0, but according to the gap Eq. (21) the factor $1 + V_0 I(\varepsilon; l, l) = 1 - \lambda N(0) \ln(\Omega_D +$

$\sqrt{\Omega_D^2 + \Delta_0^2}/\Delta_0$) is 0. To obtain the speed of sound v we expand $I(\varepsilon; l, l)$ around point 2Δ and for small Q, and we keep only small-Q^2 and small-$(\omega - 2\Delta)^2$ terms. Thus, we find $v = 2\sqrt{3/5}(\hbar k_F/m)$, i.e., the speed of sound in the weak-coupling limit does not depend on the gap.

3.3 Strong-coupling limit In strong-coupling limit $\mu < 0$. To solve the gap and the number equations one can use the fact that the gap Δ_0 is small, and therefore, we can expand the number and the gap equations in terms of Δ_0 and keep the Δ_0^2 terms only. Thus, we find the gap Δ_0 and the density n in terms of the coupling constant $\lambda = V_0 k_F/(2\pi|\mu|)$ and chemical potential $|\mu| = \hbar^2 k_F^2/(2m)$:

$$1 = \frac{\lambda \pi}{2}\left(1 - \frac{3}{16}\frac{\Delta_0^2}{|\mu|^2}\right), \quad \frac{\Delta_0^2}{|\mu|^2} = \frac{8n}{k_F}. \quad (28)$$

The next step is to expand the elements of the determinant (19) for small Q and keeping only Δ_0^2 terms. Using Eqs. (28) and (19) we obtain the energy of the Goldstone collective mode $\omega(Q)$ and the sound velocity c:

$$\omega(Q) = c\hbar Q,$$
$$c = \frac{\Delta_0}{2\hbar k_F} = v_F\sqrt{\frac{n}{2k_F}} = \left(\frac{\hbar V_0 n}{4m}\right)^{1/2}. \quad (29)$$

Note that the same analytical result can be obtained using the 3×3 determinant from Ref. [12], because for small Q the contributions due to the $\widetilde{\gamma}$-terms are of order $Q\Delta_0^2$ and higher.

The second, non-trivial mode in the strong-coupling limit with dispersion $\omega(Q) = \omega_0 + v\hbar Q$ results in ω_0 very close to $2|\mu|$. Specifically, if $V_0 k_F/2\pi\varepsilon_F = 0.639$ and $\Delta_0 = 0.145|\mu|$ numerical solution gives $\omega_0 = 1.989|\mu|$. Our perturbative result for the sound velocity c in strong-coupling differs from the exact velocity because the BCS approximation for the ground-state energy in strong coupling is different when compared to the exact BCS–Bose "crossover" theory solution [16].

4 Conclusion Since (i) superconductivity due to the formation of Cooper pairs and the corresponding analogous phenomenon of BE-condensed excitons in semiconductors are described by the same equations, and (ii) the existence of a *non-trivial* solution of the BSE is similar to the existence of a non-trivial linearly-dispersive "moving Cooper pair" collective mode that can condense in superconductors, it is natural to suggest that the presence of a secondary peak in the semiconductor absorption spectrum would constitute a signature for the existence of an excitonic BEC.

An important difference between the non-trivial modes in superconductors and in semiconductors is that in the former case both Goldstone as well as the non-trivial mode are raised well above the energy gap when the bare Coulomb exchange interaction is added (Anderson mechanism), while in the latter case the exchange interaction only contributes to excitonic fine structure of excitons so that the energy shift between the non-trivial mode and the exciton peak must be roughly twice the average value of the gap function.

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