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Superconducting Pseudogap in a Boson–Fermion Model

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A boson–fermion (BF) quantum-statistical binary gas mixture model consisting of positive-energy *resonant* bosonic Cooper electron pairs (CPs) in chemical and thermal equilibrium with single unpaired electrons is presented. Two-time retarded Green functions are shown to conveniently cope and deal with nonzero center-of-mass momentum (CMM) CPs. They yield an analytic expression for a dimensionless-coupling (λ)- and temperature (T)-dependent generalized energy gap $E_g(\lambda, T)$ in the single-electron spectrum of a superconductor. This generalized gap vanishes above a specific ("depairing" or "pseudogap") temperature $T^* > T_c$, where T_c is the critical Bose–Einstein condensation (BEC) singularity associated with the BF binary mixture, but is nonzero for all T below T^* due initially to the formation of "preformed" pairs. Within the present BF model the generalized gap $E_g(\lambda, T)$ is not restricted to the underdoped high-temperature superconductors as we illustrate with BSCCO, but is also applicable in optimally-doped and even overdoped compounds for which T^* and T_c virtually coincide as in BCS theory where nonzero-CMM CPs are neglected.

KEYWORDS: cuprate superconductivity, resonant Cooper pairs, boson–fermion models, Bose–Einstein condensation, pseudogap
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1. Introduction

Unusual experimental properties in the normal state of high-temperature superconductors (HTSCs) provide indirect signatures interpretable as the opening of a so-called *pseudogap*[1] in the electronic spectrum well above the critical temperature T_c below which the material superconducts. This has given rise to an intense and challenging debate on the origin of the pseudogap phenomenon.[2] Two basic trends are found in the literature: either the pseudogap has its origin in some phenomenon different from superconductivity[3] and which may possibly be in competition with the superconducting state, or is of the same origin[4,5] in which case it might reflect the presence of pairs above T_c, i.e., so-called "preformed pairs". As they very naturally explain the pseudogap phenomenon in terms of preformed Cooper pairs (CPs) boson–fermion (BF) models[6–9] of superconductivity (SC) are being reexamined to better understand the physics of HTSCs. BF models of SC began to be studied[10] in the mid-1950s, predating even the BCS-Bogoliubov theory of SC. However, the successes of the BCS theory[11] in describing properties of traditional low-temperature SCs left BF models neglected for many years. But the discovery of cuprate SCs, and the realization that it is impossible to describe the peculiarities of cuprates within the framework of BCS model, led to a revival of some traditional SC scenarios, in particular, of BF models which posit the existence of actual bosonic CPs. As in BCS theory we assume that the subsystem of electrons attract each other when lying (in 3D) within the spherical shell $E_F - \hbar\omega_D \leq \epsilon \leq E_F + \hbar\omega_D$ about the Fermi energy E_F of the ideal Fermi gas in single-electron ϵ energy space, with $\hbar\omega_D$ the Debye-frequency energy parameter of the s-wave BCS model interaction. Here we restrict ourselves to pure s-wave CPs but higher, *viz*, d-waves, can be also accommodated (see, e.g., refs. 12–17).

Even with the simple s-wave interaction, a gas of single fermionic charges in an ionic lattice can evolve into two coexisting and dynamically interacting subsystems: single fermionic charges (namely, pairable but unpaired charged fermions) *and* individual bosonic CP entities consisting of two mutually confined electrons. The simplest grand canonical Hamiltonian describing a binary mixture of interacting fermions with bosons as suggested in refs. 6–9 has been applied in an effort to understand the properties of HTSCs. In grand canonical form the total Hamiltonian is

$$\mathcal{H} \equiv H - \mu N = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} a^{+}_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \sum_{\mathbf{K}} \mathcal{E}_{\mathbf{K}} b^{+}_{\mathbf{K}} b_{\mathbf{K}} + H_{int} \quad (1)$$

where the first two terms on the rhs of (1) are respectively the zeroth-order Hamiltonians of free (pairable but unpaired) fermions and of composite-boson CPs. Here $a^{+}_{\mathbf{k}\sigma}$ and $a_{\mathbf{k}\sigma}$ are the usual fermion creation and annihilation operators for individual electrons of momenta $\mathbf{k}$ and spin $\sigma = \uparrow$ or $\downarrow$ while $b^{+}_{\mathbf{K}}$ and $b_{\mathbf{K}}$ are postulated[18,19] to be bosonic operators associated with CPs of definite total, or center-of-mass momentum (CMM), wavevector $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$. Fermion $\xi_{\mathbf{k}} \equiv \epsilon_{\mathbf{k}} - \mu$ and boson $\mathcal{E}_{\mathbf{K}}$ energies are measured from μ and 2μ, respectively, where μ is the fermionic chemical potential introduced in (1). If L is the system size in d dimensions, processes of boson formation/disintegration are then driven by the interaction Hamiltonian[6–9]

$$H_{\rm int} \equiv \frac{f}{L^{d/2}} \sum_{\mathbf{q},\mathbf{K}} (b^{+}_{\mathbf{K}} a_{\mathbf{q}+\mathbf{K}/2\uparrow} a_{-\mathbf{q}+\mathbf{K}/2\downarrow} + b_{\mathbf{K}} a^{+}_{-\mathbf{q}+\mathbf{K}/2\downarrow} a^{+}_{\mathbf{q}+\mathbf{K}/2\uparrow}). \quad (2)$$

In this paper f is a phenomenological BF (*two*-fermion/one-boson) vertex-interaction form-factor coupling parameter chosen as a constant nonzero only in the range $E_F - \hbar\omega_D \leq \epsilon \leq E_F + \hbar\omega_D$.

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A Hamiltonian scheme similar to (1) with (2) but extended so as to include *hole* CPs along with electron CPs gives[20] precisely the BCS gap equation for all coupling V and all T as well as the zero-T condensation energy for all $V \geq 0$, provided f is identified, as in the present work, with the attractive interelectron (*four*-fermion) interaction strength V of the s-wave BCS model interaction through the relation[18,19] $f = \sqrt{2\hbar\omega_D V}$. refs. 18–20 ignore nonzero-CMM CPs in the interaction Hamiltonian H_{int} but not in the unperturbed Hamiltonian, while they are completely ignored in the entire Hamiltonian in the original BCS theory. We also note that the "two-fermion/one-boson" interaction form (2) used here and familiar "four-fermion" interaction employed by BCS[11] originate from two distinct Feynman diagrams respectively referred to in Fig. 1 of ref. 7 as the s- and t-channels in the language of particle physics. Here, nonzero-CMM CPs are not ignored in (2) and we show how two-time Green functions can deal with such pairs. This then allows deriving an analytical expression for both the pseudogap and superconducting gap energies originating from the BF binary mixture model (1) and (2) based on bosonic CPs having the gapped excitation spectrum given by (4) below and which describes *resonant* CPs as advocated originally by Schafroth[10] in 1954.

2. Gapped Resonant Cooper Pairs

In refs. 21–23 it was suggested that if electron CPs are considered as composite bosons shifted in energy at zero $\mathbf{K}$ from the Fermi energy E_F by a *positive* gap it then becomes possible to exhibit *two* characteristic temperatures, T^* and T_c. First, a depairing temperature T^* below which the electronic chemical potential $\mu(\lambda, T)$ first dips below the E_F associated with interactionless electrons: below this T^* the first CPs begin to appear in the system. Here, we introduce the usual dimensionless coupling parameter $\lambda \equiv N(0)V$ with $N(0)$ the electronic density of states (DOS) for each spin at the Fermi surface. The condition $E_F - \mu(\lambda, T^*) = 0$ then yields the T^* below which a transition occurs from the normal state with *no* composite bosons to one *with* such bosons, i.e., at temperatures below T^* the system of attractively-interacting fermions becomes a binary BF mixture, albeit *incoherent*. Second, the BEC temperature T_c at which a singularity first occurs in the total number density of bosons

$$n_B(\lambda, T_c) \equiv L^{-d} \sum_{\mathbf{K}} \left[\exp\left(\frac{\Omega_{\mathbf{K}}}{k_B T_c} \right) - 1 \right]^{-1} \tag{3}$$

where $\Omega_{\mathbf{K}}$ is essentially the boson energy $\mathcal{E}_{\mathbf{K}}$ appearing in (1) but *renormalized*[21] by the presence of BF interactions as embodied in (2). We stress that determining two distinct temperatures T^* and T_c, where $T_c < T^*$, that behave qualitatively as is observed in high-T_c cuprates, was possible owing *only* to assuming in (1) a boson spectrum such as

$$\mathcal{E}_{\mathbf{K}} \equiv 2E_F + 2\Delta(\lambda) + \varepsilon_{\mathbf{K}} \tag{4}$$

where $\varepsilon_{\mathbf{K}}$ is a nonnegative CP excitation energy that vanishes when $K = 0$. Thus, in (4) $\mathcal{E}_{\mathbf{K}}$ is *higher* than the total energy $2E_F$ of two noninteracting individual electrons.

How is a boson energy $\mathcal{E}_{\mathbf{K}}$ given by (4) feasible and how do attractively-interacting electrons confined into bosonic CPs with a spectrum such as (4) lead to an energy lower than that associated with the system of interactionless fermions? To answer the first part of this question we refer to refs. 24 and 25 (for a recent review see ref. 26) where using the Bethe–Salpeter integral equation in the ladder approximation for *two-particle* and *two-hole* coupled wavefunctions, in either 3D[24] or 2D,[25] yielded (in leading order of the CMM wavenumber K) a linearly-dispersive $\varepsilon_{\mathbf{K}}$ *above* the total energy $2E_F$ of two interactionless electrons and on top of a positive bosonic gap $2\Delta(\lambda)$, where

$$\Delta(\lambda) = \frac{\hbar\omega_D}{\sinh(1/\lambda)} \xrightarrow[\lambda \to 0]{} 2\hbar\omega_D \exp\left(-\frac{1}{\lambda}\right) \tag{5}$$

is the usual $T = 0$ BCS fermionic energy gap for the BCS model interelectronic interaction. Traven[27,28] has also found pair excitations in the ground state of a 2D attractive Fermi gas by going beyond the standard random phase approximation (RPA). Besides a long-wavelength soundlike collective mode, he finds pair excitations with energy $\geq 2\Delta$ in the weak-coupling-limit, where 2Δ is the threshold for the decay into two fermionic quasiparticles. A year later similar results were also reported[29] for an attractive δ-function interfermion interaction in the 1D fermion gas.

It might be suspected that excitations with a positive gap $2\Delta(\lambda)$ would increase the energy contribution from the second term on the rhs of (1), i.e., from free bosons, and therefore that their existence would seem to be energetically unfavorable. This suspicion, however, is not borne out on comparing the energies of the BF mixture described by (1) and (2) and that of interactionless fermions since it was found[30] that introducing an attractive interaction between electrons in the gas of electrons leads to the formation of a new type of lower-energy gas mixture state with bosonic excitations *above* the Fermi sea of unpaired electrons. Competition between electrons to occupy energy levels below E_F so as to minimize the volume of the Fermi sea expels from the sea attractively-interacting charge carrier levels and raises them above E_F. These raised charge carriers appear outside the Fermi sea and are mutually confined into positive-energy resonant CPs. Processes of pair formation and their subsequent disintegration into two unpaired electrons are driven by (2) and were crucial in ref. 30 in forming the BF mixture state with positive-energy bosonic excitations. Separation of the initial attractively-interacting-fermion system into bosons and fermions with spectra shifted with respect to each other by the coupling-dependent, positive-energy gap, first anticipated in ref. 24, is a new ingredient in the BF model (1) introduced in refs. 21–23, 30.

3. Two-Time Green Functions

The nature of a pseudogap within the present BF binary gas mixture model, whether this phenomenon is of the same origin as superconductivity or not, and what distribution of free carriers occurs below some $T^* > T_c$, can be addressed by starting from the T-dependent occupation number of unpaired electrons $n_{\mathbf{k},\sigma} \equiv \langle a^+_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} \rangle$ in a state with momentum $\mathbf{k}$ and spin σ, where single-angular brackets $\langle X \rangle$ of an operator X is the T-dependent thermal average over a grand canonical Hamiltonian $\mathcal{H} \equiv H - \mu N$. The c-numbers $n_{\mathbf{k},\sigma}$ can then be found, e.g., from the infinite chain of equations

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle A(t)\mid B(t')\rangle\rangle = \mathrm{i}\delta(t-t')\langle[A(t),B(t')]_\eta\rangle + \langle\langle[A(t),H]\mid B(t')\rangle\rangle \tag{6}$$

for two-time retarded Green functions, designated with double-angular brackets, as defined in ref. 31 eq. (2.1b) for dynamical operators $a_{\mathbf{k}\uparrow}(t)$ and $a^+_{\mathbf{k}'\uparrow}(t')$ at times t and t'. Square brackets $[A,B]_\eta \equiv AB + \eta BA$ denote the commutator ($\eta = -1$) or anticommutator ($\eta = +1$) of any two operators A and B. In this formalism any dynamical operator $X(t)$ is in the Heisenberg representation, i.e., is of the form $X(t) = \exp(\mathrm{i}\mathcal{H}t)X\exp(-\mathrm{i}\mathcal{H}t)$. The Fourier transform $\langle\langle A\mid B\rangle\rangle_\omega$ of $\langle\langle A(t)\mid B(t)\rangle\rangle$ in ω satisfies the chain of equations [ref. 21, eq. (A2)], namely

$$\hbar\omega\langle\langle A\mid B\rangle\rangle_\omega = \langle[A,B]_\eta\rangle_{\mathcal{H}} + \langle\langle[A,\mathcal{H}]_-\mid B\rangle\rangle_\omega. \tag{7}$$

Knowing $\langle\langle a_{\mathbf{k}\alpha}\mid a^+_{\mathbf{k}'\alpha}\rangle\rangle_\omega$ one can find expressions for the corresponding thermal-average values $\langle a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}\rangle_{\mathcal{H}}$ from the so-called spectral density $J(\omega)$ (ref. 31. p. 78). Choosing in (7) *first* $A \equiv a_{\mathbf{k}\uparrow}$ and *then* $A \equiv a^+_{\mathbf{k}\downarrow}$ and setting B equal to $a^+_{\mathbf{k}'\uparrow}$, one obtains after some algebra [refs. 21 and 22, eq. (A6)]

$$(\hbar\omega - \epsilon_{\mathbf{k}} + \mu)\langle\langle a_{\mathbf{k}\uparrow}\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega = \delta_{\mathbf{k}\mathbf{k}'} + \frac{f^2}{L^d}\sum_{\mathbf{K},\mathbf{Q}}\frac{\langle b_{\mathbf{K}}\rangle\langle b^+_{\mathbf{Q}}\rangle}{\hbar\omega + \epsilon_{-\mathbf{k}+\mathbf{K}} - \mu}\langle\langle a_{\mathbf{k}-\mathbf{K}+\mathbf{Q}\uparrow}\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega \tag{8}$$

where we have set $\eta = +1$ in (7) signifying anti-commutation relations for fermion operators and where the b, b^+ operators are defined in (2). Furthermore, higher-order Green functions $\langle\langle BC\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle$ coming from the last term on the rhs of (7) are put into the form

$$\langle\langle BC\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega = \langle B\rangle\langle\langle C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega + \langle\langle(B-\langle B\rangle)C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega. \tag{9}$$

We note that the mean values $\langle b_{\mathbf{K}}\rangle$ and $\langle b^+_{\mathbf{K}}\rangle$ are identically zero for a pure Bose gas described by the Hamiltonian $H^0_{\mathrm{B}} \equiv \sum_{\mathbf{K}} E^0_{\mathbf{K}} b^+_{\mathbf{K}} b_{\mathbf{K}}$. This is a result of the degeneracy of a single statistical state associated with commuting $N_{\mathrm{B}} \equiv \sum_{\mathbf{K}} b^+_{\mathbf{K}} b_{\mathbf{K}}$ and H^0_{B} operators implying conservation of the number of Bose particles. This degeneracy can be removed by adding to H^0_{B} a term like $\nu(b_{\mathbf{0}} + b^+_{\mathbf{0}})$ to give $H^0_{B,\nu} \equiv H^0_{\mathrm{B}} + \nu(b_{\mathbf{0}} + b^+_{\mathbf{0}})$ with an infinitesimal ν eventually taken to zero. The added terms in $b_{\mathbf{0}}$ and $b^+_{\mathbf{0}}$ violate commutativity between N_{B} and $H^0_{B,\nu}$. Then, below a critical T_{c}, nonzero $\langle b_{\mathbf{0}}\rangle$ and $\langle b^+_{\mathbf{0}}\rangle$ signal the emergence of a new order in the pure Bose gas.[32] In the BF mixture described by (1) $[N_{\mathrm{B}}, H] \neq 0$ implying nonconstant N_{B} that varies with temperature T and coupling λ. Thus, (2) breaks the degeneracy associated with the number and as well as total momentum conservation laws and leads, in particular, to nonzero $\langle b_{\mathbf{K}}\rangle$ and $\langle b^+_{\mathbf{K}}\rangle$ signalling the emergent BF mixture state. Thus, contributions in (9) from terms proportional to $\langle b_{\mathbf{K}}\rangle$ and $\langle b^+_{\mathbf{K}}\rangle$ are most important. On the other hand, the total number operator

$$N = \sum_{\mathbf{k},\sigma} a^+_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} + 2\sum_{\mathbf{K}} b^+_{\mathbf{K}} b_{\mathbf{K}} \tag{10}$$

which includes both the numbers of unpaired electrons and of twice the bosons, does commute with (1)[9] and is thus an invariant of motion for the BF mixture state.

Note that within all so-called "first-order theories" (ref. 32, p. 100) higher-order Green functions of the type $\langle\langle BC\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ are put in a form of linear combinations of the first-order Green functions $\langle\langle C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ and $\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$, *viz.*, via

$$\langle\langle BC\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega = \langle B\rangle\langle\langle C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega + \langle C\rangle\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega. \tag{11}$$

There are no terms containing functions such as $\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ in (9) where $B = b_{\mathbf{K}}$ or $b^+_{\mathbf{K}}$, i.e., a pure boson operator. If the vertex BF interaction parameter f in (2) is a small parameter and the chain (7) is applied to obtain an equation for $\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ one finds that this is proportional to f^2, i.e., $\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ contains an extra f^2 with respect to terms like $\langle\langle C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ with $C = a_{\mathbf{k}\alpha}$ or $a^+_{\mathbf{k}\alpha}$ being a pure fermion operator. Formally, the smallness of f justifies the absence of terms like $\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ in (9). There is another reason for ignoring the last term on the rhs of (11). In fact, for a system conserving the total number of fermions, averages over all products $a^+a^+a^+\cdots aa$ of *unequal* numbers of creation $a^+_{\mathbf{k}\alpha}$ and annihilation $a_{\mathbf{k}\alpha}$ operators must be zero (ref. 32, p. 8). This is exact for a system with a single kind of particle. For the BF mixture defined by the Hamiltonian (1) and (2) the total number of fermions, which from (10) is as a sum of pairable (but unpaired) fermions and paired ones, is evidently conserved. However, *nonzero* averages like $\langle a_{\mathbf{k}\alpha}\rangle$ and $\langle a^+_{\mathbf{k}\alpha}\rangle$ appear due to the cross terms from (2) in the total Hamiltonian. Recalling the identity

$$\langle[C,\mathcal{H}]\rangle_{\mathcal{H}} \equiv \mathrm{Tr}\{(C\mathcal{H} - \mathcal{H}C)\exp(-\beta\mathcal{H})\} = 0 \tag{12}$$

where C is any dynamical operator (chosen here to be $a_{\mathbf{k}\alpha}$ or $a^+_{\mathbf{k}\alpha}$) and the thermal average $\langle[C,\mathcal{H}]\rangle_{\mathcal{H}}$ of the commutator $[C,\mathcal{H}]$ is performed over the Hamiltonian $\mathcal{H} \equiv H - \mu N$, it is not difficult to see that the cross terms given by (2) lead to the appearance of $\langle a_{\mathbf{k}\alpha}\rangle$ and $\langle a^+_{\mathbf{k}\alpha}\rangle$ which turn out to be $O(f^3)$. Therefore, at least for a mixture with *weakly* interacting boson and fermion subsystems we expect that the contribution from $\langle C\rangle\langle\langle B\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ in (11) will be $O(f^5)$. Finally, terms of the type $\langle\langle(B-\langle B\rangle)C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ on the rhs of (9) are considered small contributions and hence neglected in getting (8). The last approximation is similar to Tyablikov's [ref. 33, p. 258, eq. (32.7)] random phase approximation (RPA) which underlies *all* first-order theories (ref. 32 pp. 108 ff.). That is, if in considering higher than first-order Green functions one assumes that the phases of pure boson B and pure fermion C operators vary independently, then by averaging over H in (9) to obtain $\langle\langle(B-\langle B\rangle)C\mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ this vanishes precisely because of the factor $B-\langle B\rangle$.

4. Boson Number Density

Note that k and K in (13) are wavenumbers corresponding to fermions and bosons, respectively. The magnitude of boson center-of-mass momentum wavenumbers K are defined by the energy scale of 2e-bosonic excitations which in turn can be expected to be much less than the highest energy scale E_{F} in the present model. In particular, the condensation energy per pair of electrons describing the decrease in the energy-per-particle and which occurs upon the transition from the normal into the superconducting state, may be considered a suitable energy scale for the 2e-CPs.

For estimates one may therefore assume that $\hbar^2 K_{\rm max}^2/2m_{\rm B} \ll \hbar^2 k_{\rm F}^2/2m \equiv E_{\rm F}$ where $K_{\rm max}$ is the maximum value of 2e-CP wavenumbers K and $m_{\rm B}$ is approximately twice of the effective electron mass m. The maximum value $K_{\rm max}$ is that critical value of K beyond which breakups or depairings of 2e-CPs occurs.[34,35] Hence, K can be taken as much less than $k_{\rm F}$. As to fermion wavevectors $\mathbf{k}$ and $\mathbf{k}'$, their magnitude is of order $k_{\rm F}$. Therefore, one may assume that $k \gg K$ in (13), thanks to which $\epsilon_{-\mathbf{k}+\mathbf{K}} \simeq \epsilon_{-\mathbf{k}}$ and $\langle\langle a_{\mathbf{k}-\mathbf{K}+\mathbf{Q}\uparrow} \mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega \simeq \langle\langle a_{\mathbf{k}\uparrow} \mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega$ allowing one to factor all terms except $\langle b_{\mathbf{K}}\rangle\langle b^+_{\mathbf{Q}}\rangle$ outside the summation in (8). Collecting in (8) the terms with $\delta_{\mathbf{kk}'}$ yields

$$\langle\langle a_{\mathbf{k}\uparrow} \mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle_\omega \simeq \delta_{\mathbf{kk}'}\left(\hbar\omega - \epsilon_{\mathbf{k}} + \mu - \frac{f^2}{L^d}\sum_{\mathbf{K}}\frac{\langle b_{\mathbf{K}}\rangle\langle b^+_{\mathbf{K}}\rangle}{\hbar\omega + \epsilon_{-\mathbf{k}+\mathbf{K}} - \mu}\right)^{-1}. \tag{13}$$

The rhs of (13) is exactly as in eq. (A·7) of ref. 21. Note that if translational symmetry holds functions like $\langle\langle a_{\mathbf{k}\uparrow} \mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle$ and $\langle\langle a_{\mathbf{k}-\mathbf{K}+\mathbf{Q}\uparrow} \mid a^+_{\mathbf{k}'\uparrow}\rangle\rangle$ turn out to be diagonal as in lowest-order two-time Green functions (ref. 33, p. 244).

In (13) we assume that $\langle b_{\mathbf{K}}\rangle\langle b^+_{\mathbf{K}}\rangle \simeq \langle b^+_{\mathbf{K}} b_{\mathbf{K}}\rangle = \langle n_{B\mathbf{K}}\rangle$ for any $K \geq 0$ with $\langle n_{B\mathbf{K}}\rangle$ the number of bosons with CMM $\mathbf{K}$. This approximation, the justification of which is given in Appendix A, holds *only for the particles obeying Bose statistics*, i.e., when $\langle L^{-d} b^+_{\mathbf{K}} b_{\mathbf{K}}\rangle$ and $\langle L^{-d} b_{\mathbf{K}} b^+_{\mathbf{K}}\rangle$ are macroscopic numbers: their difference vanishes asymptotically as $\sim L^{-d}$ making the effects associated with the non-commutativity of $b_{\mathbf{K}}$ and $b^+_{\mathbf{K}}$ unimportant in this limit. Thus in (13) we may insert

$$L^{-d}\sum_{\mathbf{K}}\langle b_{\mathbf{K}}\rangle\langle b^+_{\mathbf{K}}\rangle \simeq L^{-d}\sum_{\mathbf{K}}\langle n_{B\mathbf{K}}\rangle \equiv n_{\rm B}(\lambda, T) \tag{14}$$

where $n_{\rm B}(\lambda, T)$ is the *net* number density (3) of CPs in the BF mixture at any T. It is exactly zero for all $T \geq T^*$ and begins to differ from zero as one lowers the temperature T below T^*. Arguably, $\langle n_{B\mathbf{K}}\rangle \equiv \langle b^+_{\mathbf{K}} b_{\mathbf{K}}\rangle$ decreases sharply with wavenumber K thus corroborating the assumption $k \gg K$ used in (13).

The value of $n_{\rm B}(\lambda, T)$ was related in ref. 21, eq. (20) with the magnitude of $E_{\rm F} - \mu(\lambda, T)$ driven by bosonization. Thus, from eq. (20) in ref. 21 one finally has

$$n_{\rm B}(\lambda, T) \simeq N(E_{\rm F})[E_{\rm F} - \mu(\lambda, T)] \tag{15}$$

for the net boson number density, condensed and non-condensed.

5. Generalized Gap

Relating the phenomenological BF vertex ("*two*-electron/one-boson") coupling parameter f with the attractive interelectron ("*four*-electron") BCS interaction model[11] strength V via $f = \sqrt{2\hbar\omega_{\rm D} V}$[18–20] and recalling that $\lambda \equiv N(E_{\rm F})V$ makes (13) become

$$\langle\langle a_{\mathbf{k}\uparrow} \mid a^+_{\mathbf{k}'}\rangle\rangle_\omega \simeq \delta_{\mathbf{kk}'}\frac{\hbar\omega + \xi_{\mathbf{k}}}{2E_{\mathbf{k}}}\left(\frac{1}{\hbar\omega - E_{\mathbf{k}}} - \frac{1}{\hbar\omega + E_{\mathbf{k}}}\right) \tag{16}$$

since $\epsilon_{\mathbf{k}} = \epsilon_{-\mathbf{k}}$ and where

$$\sqrt{\xi^2_{\mathbf{k}} + E^2_{\rm g}(\lambda, T)} \equiv E_{\mathbf{k}}(\lambda, T) \tag{17}$$

with the generalized gap $E_{\rm g}(\lambda, T)$ defined as

$$E_{\rm g}(\lambda, T) \equiv \sqrt{2\lambda\hbar\omega_{\rm D}[E_{\rm F} - \mu(\lambda, T)]} \tag{18}$$

and where as before $\xi_{\mathbf{k}} \equiv \epsilon_{\mathbf{k}} - \mu$. Preliminarily establishing the spectral density [ref. 31, eq. (3.25)] and then applying it to calculate $\langle a^+_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma}\rangle$ gives

$$n_{\mathbf{k}} \equiv \sum_\sigma \langle a^+_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma}\rangle = \frac{1}{2}\left[1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{E_{\mathbf{k}}}{2k_{\rm B}T}\right)\right]. \tag{19}$$

The departure of $\mu(\lambda, T)$ from $E_{\rm F}$ in (18) is found as in deriving (13) by recalling that $k \gg K$ and repeating the calculation reported in ref. 21, eq. (32). This finally gives

$$E_{\rm F} - \mu(\lambda, T) = -\Delta(\lambda) + \frac{\lambda\hbar\omega_{\rm D}}{2}\int_{-\hbar\omega_{\rm D}}^{\hbar\omega_{\rm D}}\frac{{\rm d}x}{\sqrt{x^2 + E^2_{\rm g}(\lambda, T)}} \times \tanh\frac{\sqrt{x^2 + E^2_{\rm g}(\lambda, T)}}{2k_{\rm B}T} \tag{20}$$

where $\Delta(\lambda)$ is given by (5).

If (15) and (18) are combined, one immediately arrives at the principal result of this paper

$$E_{\rm g}(\lambda, T) = \sqrt{2\hbar\omega_{\rm D} V n_{\rm B}(\lambda, T)} \equiv f\sqrt{n_{\rm B}(\lambda, T)}. \tag{21}$$

This relates the pseudogap order parameter $E_{\rm g}(\lambda, T)$ very naturally with the total number density of electron pairs $n_{\rm B}(\lambda, T)$. Thus, from (17) it is $E_{\rm g}(\lambda, T)$ that determines the single-fermion spectrum $E_{\mathbf{k}}(\lambda, T)$. Recalling that $\Delta(\lambda, T)$ is the ordinary BCS energy gap and $n_0(\lambda, T)$ the BEC condensate density associated with the zero-CMM state, the relation that unifies BCS with BEC, namely

$$\Delta(\lambda, T) \propto \sqrt{n_0(\lambda, T)} \tag{22}$$

first seems to have appeared in ref. 6. It later resurfaced in the phenomenological BF BEC model that Friedberg and Lee[7,8] applied to cuprate superconductors. With just *one* adjustable parameter (the ratio of perpendicular to planar boson masses) they fitted[8] cuprate $T_{\rm c}/T_{\rm F}$ data quite well. The ratio turned out to be 66,560 — just under the 10^5 anisotropy ratio reported[36] almost contemporaneously for $Bi_{2+x}Sr_{2-y}CuO_{6\pm\delta}$. Later on, the *precise* connection with BCS theory was finally established in refs. 18–20. Clearly, as $n_{\rm B}(\lambda, T)$ in (21) is larger than $n_0(\lambda, T)$ of (22) exactly by the total number density of *excited* or nonzero-CMM pairs, (21) indeed generalizes the BCS–BEC relation (22) known for at least 20 years. CPs begin to appear not exactly at and below $T_{\rm c}$ as in BCS theory, but at temperatures much above the $T_{\rm c}$ of the BF mixture BEC. It is perhaps for this reason that one refers to "preformed" CPs. Therefore, being proportional as in (21) to the square root of the number density of *all* of the bosons, the generalized gap $E_{\rm g}(\lambda, T)$ in the single-fermion spectrum emerges at temperatures much higher than $T_{\rm c}$.

Equation (18) connecting the generalized gap $E_{\rm g}(\lambda, T)$ with the departure of $\mu(\lambda, T)$ from $E_{\rm F}$ due to bosonization, can be rewritten as

$$E_{\rm F} - \mu(\lambda, T) = \frac{E^2_{\rm g}(\lambda, T)}{2\lambda\hbar\omega_{\rm D}} \tag{23}$$

which when inserted in (20) gives the implicit equation for $E_{\rm g}(\lambda, T)$ as

$$\frac{E^2_{\rm g}(\lambda, T)}{2\lambda\hbar\omega_{\rm D}} = -\Delta(\lambda) + \frac{\lambda\hbar\omega_{\rm D}}{2}\int_{-\hbar\omega_{\rm D}}^{\hbar\omega_{\rm D}}\frac{dx}{\sqrt{x^2 + E^2_{\rm g}(\lambda, T)}}$$

$$\times \tanh \frac{\sqrt{x^2 + E_g^2(\lambda, T)}}{2k_B T}. \qquad (24)$$

According to (23), as the chemical potential $\mu(\lambda, T)$ of attractively interacting fermions becomes less than E_F, a real nonzero generalized gap $E_g(\lambda, T)$ satisfying (24) appears in the spectrum of single fermions. Note that (23) and hence (24) have no real solution for $T > T^*$. If $\mu(\lambda, T) > E_F$ in (23) then a purely-imaginary $E_g(\lambda, T)$ emerges. However, a separation of the original many-fermion system into bosons and fermions does not occur until $E_F - \mu(\lambda, T) \geq 0$.

Valid for any $T \leq T^*$ and any coupling λ (24) defines $E_g(\lambda, T)$ in terms of the boson gap separation $2\Delta(\lambda)$ between the Fermi sea and the bottom of the resonant CP band at $K = 0$. Introducing $\Delta(\lambda)$ was decisive[21] in understanding the reason why two specific temperatures, T^* and T_c with $T^* > T_c$, appear in the picture. In particular, the depairing event at T^* was explained as a result of resonant CPs occupying energy levels above the Fermi sea by $2\Delta(\lambda)$. Then putting $E_F - \mu(\lambda, T^*) = 0$ (see detailed discussion in ref. 21, p. 104506-5) connects T^* directly with $\Delta(\lambda)$ via

$$\Delta(\lambda) = \frac{\lambda \hbar\omega_D}{2} \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{dx}{x} \tanh \frac{x}{2k_B T^*}. \qquad (25)$$

If (25) is inserted in (24) one gets the equation

$$\frac{E_g^2(\lambda, T)}{(\lambda\hbar\omega_D)^2} + \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{dx}{x} \tanh \frac{x}{2k_B T^*} - \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{dx}{\sqrt{x^2 + E_g^2(\lambda, T)}} \tanh \frac{\sqrt{x^2 + E_g^2(\lambda, T)}}{2k_B T} = 0 \qquad (26)$$

relating the coupling- and temperature-dependent generalized gap $E_g(\lambda, T)$ with the T^* at which the pseudogap opens in the single-particle fermion spectrum. Equation (26) contains, instead of the theoretical $\Delta(\lambda)$ given by (5), an experimentally observable $E_g(\lambda, T)$ and T^* which can thereby be taken from experiment as a fitting parameter. In particular, based on experimental data on $E_g(\lambda, T)$ and T^*, valuable information on the magnitudes of λ and $\hbar\omega_D$ can be extracted from (26). For estimation purposes, assuming $\hbar\omega_D$ to be in the range 50–80 meV[37] and using the empirical values of $E_g(\lambda, T) = 38$ meV and $T^*/T_c \simeq 3.5$ reported in ref. 38 for the $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Bi2212 or BSCCO) compound with $T_c = 74$ K, calculations based on (26) give $\lambda \sim 0.8$ to 1.3 with the smaller λ corresponding to larger $\hbar\omega_D$.

Equation (24) becomes especially simple at $T = 0$ when

$$\frac{E_g^2(0)}{(\lambda\hbar\omega_D)^2} - \ln\left[1 + 2\left(\frac{\hbar\omega_D}{E_g(0)}\right)^2 + 2\left(\frac{\hbar\omega_D}{E_g(0)}\right)\sqrt{1 + \left(\frac{\hbar\omega_D}{E_g(0)}\right)^2}\right] = \int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{dx}{x} \tanh \frac{x}{2k_B T^*} \qquad (27)$$

with $E_g(0) \equiv E_g(\lambda, 0)$ the $T = 0$ generalized gap. Clearly, (17) formally resembles the gapped single-fermion energies $\sqrt{\xi_{\mathbf{k}}^2 + \Delta^2(\lambda, T)}$ of the BCS theory.[11] However, the BCS gap $\Delta(\lambda, T)$ and the generalized gap $E_g(\lambda, T)$ are of entirely distinct origins. Once formed at and below T_c CPs in the BCS model are allowed to go only over all paired-two-particle states. The breakup of a CP into two unpaired fermions requires an energy $2\Delta(\lambda, T)$ which must be supplied to the system to remove a CP from the subsystem of CPs. Namely, owing to the mutual transitions between CPs the BCS state happens energetically favorable compared to that of the assembly of attractively-interacting single fermions, ref. 11, eq. (2.42). In sharp contrast, however, what lowers energy of the BF state compared to that of the assembly of pairable fermions are the continual transitions between CPs and unpaired fermions. These transitions are driven by the term (2) in (1). Occupation of any single fermion k state by an electron not participating in those continual transitions, blocks many of CP pair-to-unpaired-2e (and *vice versa*) transitions that would otherwise occur. Such blockage of transitions raises the total system energy as was shown in ref. 30, eq. (31); this renormalizes the single-particle spectrum and gives rise to the generalized gap $E_g(\lambda, T)$.

The pseudogap in the present model appears as an actual energy gap in the single-fermion spectrum. It appears already at temperatures higher than T_c. This contrasts with the gap in BCS theory which emerges exactly at and below T_c. It would thus be interesting to find experimental clues as to whether physical properties such as charge transport are associated with unpaired fermions whose density of states is suppressed in the temperature range $T_c \leq T \leq T^*$ or with charged 2e bosons.

6. Illustration with BSCCO and Discussion

In Fig. 1 we plot the generalized gap $E_g(\lambda, T)$ (in units of E_F) as a function of reduced temperature T/T_F as obtained from (24) where (5) was used for $\Delta(\lambda)$. From the Table on p. 596 of ref. 39 T_F in quasi-2D superconductors ranges from 510 to 3150 K which for a median value for cuprates of $\hbar\omega_D/k_B = 400$ K[40] gives a Debye-to-Fermi temperature ratio $\hbar\omega_D/E_F$ in the range 0.13 to 0.78. For illustration, the numerical results shown in Fig. 1 correspond to a typical value of $\hbar\omega_D/E_F = 0.35$ within the stated range. The qualitative behavior in T of $E_g(\lambda, T)/E_F$ does not change radically over the range 0.13 to 0.78 of $\hbar\omega_D/E_F$.

Points on the curves in this figure refer to the values of T_c/T_F, namely 0.036 and 0.046 for $\lambda = 0.6$ (full curve) and

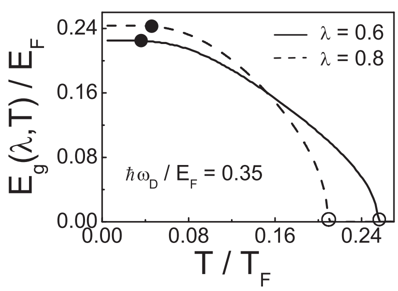

Fig. 1. Dimensionless generalized gap $E_g(\lambda, T)/E_F$ as function of dimensionless temperature T/T_F for $\lambda = 0.6$ (full curve) and 0.8 (dashed curve), both for the typical cuprate value of $\hbar\omega_D/E_F = 0.35$. Dots on curves refer to theoretical T_c/T_F found by applying the expression for T_c established for 2D superconductors [ref. 21, eq. (35)]. Open circles locate T^* values. Curves are otherwise qualitatively the same for any $\hbar\omega_D/E_F$ in the range 0.13 to 0.78.

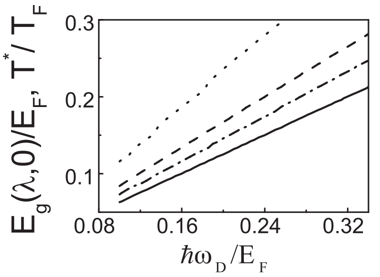

Fig. 2. Shown as functions of $\hbar\omega_D/E_F$ are zero-temperature gap $E_g(0)/E_F$ (full curve for $\lambda = 0.4$, dashed for $\lambda = 0.6$) found from (24) and T^*/T_F extracted from (25) (dotted curve for $\lambda = 0.4$ and dot-dashed for $\lambda = 0.6$) where (5) is inserted in the lhs of (25).

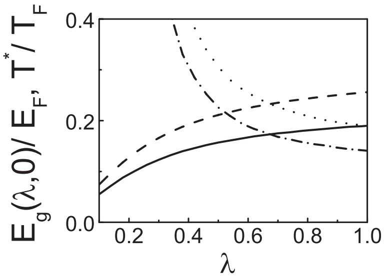

Fig. 3. Shown as functions of coupling parameter λ are zero-temperature generalized gap $E_g(0)/E_F$ (full curve for $\hbar\omega_D/E_F = 0.26$ and dashed for $\hbar\omega_D/E_F = 0.35$) and T^*/T_F (dotted curve for $\hbar\omega_D/E_F = 0.35$ and dot-dashed for $\hbar\omega_D/E_F = 0.26$). Note that $E_g(0)$ rises while T^* falls with increasing λ, and combining to render $2E_g(0)/k_BT^*$ insensitive to changes in λ at larger λs.

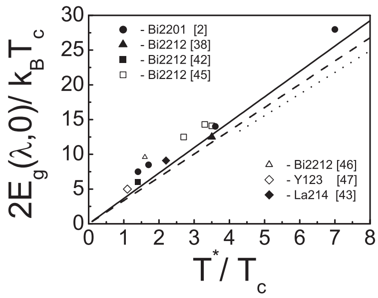

Fig. 4. Shown as functions of T^*/T_c is the generalized-gap-to-T_c ratio $2E_g(\lambda,0)/k_BT_c$ for $\lambda = 0.86$, 1.2, and 3. Increasing λ increases the curve slope, approaching a value of about 3.67 which keeps roughly constant at 3.2 to 3.7 for all $\lambda > 0.8$. Symbols in figure refer to experimental data points for various Bi-based and for La_2CuO_4 (Li124) and $YBa_2Cu_3O_{7-\delta}$ (Y123) cuprate compounds reproduced from Fig. 5 of ref. 2 where data are collected from different sources cited therein.

0.8 (dashed curve), respectively, are found from the expression for T_c established in[21] for quasi-2D superconductors [ref. 21, eq. (35)]. Evidently, the generalized gap does not close at and above T_c as does the coherent-state gap, say $\Delta(\lambda, T)$, predicted by BCS theory for ordinary superconductors. Instead, upon further heating it evolves into a well-defined pseudogap which over an appreciably broad range of temperatures is roughly of the same order of magnitude as the superconducting gap itself. The pseudogap gradually fills up and finally vanishes above $T^*/T_F = 0.255$ and 0.209, respectively, where the $E_g(\lambda, T)/E_F$ curves dip below the horizontal axis. As to the value of the coherent-state gap at temperatures below T_c, it remains nearly unchanged as T dips below T_c. This contrasts with the gap associated with BCS superconductors which is precisely zero for all $T \geq T_c$. The behavior shown in the figure is seen to agree qualitatively with tunneling conductance measurements (Fig. 3, ref. 2) of the overdoped $Bi_2Sr_2CuO_{6+\delta}$ (Bi2201) compound.

With the generalized gap $E_g(\lambda, T)$ below T_c and $E_g(0)$ the value of the gap at $T = 0$, numerical calculations show that for *any fixed* BCS coupling parameter λ, increasing $\hbar\omega_D/E_F$ always leads to a linear increase in both $E_g(0)$ and T^* as seen in Fig. 2 for different λ. Full and single-dashed curves depict the value $E_g(0)$ (in E_F units) for $\lambda = 0.4$ and 0.6, respectively. Dimensionless depairing temperatures T^*/T_F vs $\hbar\omega_D/E_F$ for the present BF model are indicated dotted ($\lambda = 0.4$) and dot-dashed ($\lambda = 0.6$) curves.

The linear rise of both $E_g(0)$ and T^* as $\hbar\omega_D/E_F$ increases is clearly evident in this figure. The magnitude of $E_g(0)$ increases with λ in contrast with the depairing temperature T^* which decreases as λ increases (see discussion in ref. 21, p. 104506-7). Indeed, no matter how large T is, any attractive interaction between electrons leads to nonzero expectation of pair formations. As was established in ref. 21, resonant CPs energetically closer to the Fermi surface, namely those that according to (5) formed with smaller λ, are better protected against breakups and thus survive over a wider temperature range. In contrast with common wisdom[41] T^*, interpreted as the temperature below which the first paired states appear, is here expected to be higher in superconductors with smaller λ than in those with larger λ.

From Fig. 2 the ratio $2E_g(0)/k_BT^*$ (defined in analogy with the emblematic dimensionless gap-to-T_c ratio $2\Delta(0)/k_BT_c \simeq 3.53$ of BCS theory with T^* replacing T_c) does not depend on the energy width $2\hbar\omega_D$ of the interaction shell $E_F - \hbar\omega_D \leq \epsilon \leq E_F + \hbar\omega_D$ as also occurs with the BCS ratio $2\Delta(0)/k_BT_c$. However, as seen in Fig. 3, for fixed $\hbar\omega_D/E_F$ with 0.26 (full and dot-dashed curves, respectively for $E_g(0)$ and T^*) and 0.35 (dashed and dotted curves, respectively for $E_g(0)$ and T^*) the response of $E_g(0)$ and T^* to changes in λ are qualitatively different as $E_g(0)$ increases as λ increases whereas T^* decreases within the overall interval of variation of λ. For small λ the decrease in T^* is rapid but for larger λ, T^* becomes less sensitive to changes in λ.[21]

The dimensionless ratio $2E_g(0)/k_BT^*$ for the small λ is strongly λ-dependent but, as is evident below in Fig. 4, as λ increases past about 0.8 the ratio becomes largely independent of λ. Results of numerous calculations are shown in Fig. 4, namely that for *all* λ the magnitude of $2E_g(0)/k_BT_c$ rises proportionally with T^*/T_c. For *any* variation of T^*/T_c the slope of a graph $2E_g(0)/k_BT_c$ vs T^*/T_c in Fig. 4 remains unchanged. The proportionality between $2E_g(0)/k_BT_c$ and T^*/T_c implies that the ratio $2E_g(0)/k_BT^*$ is a function of λ *only*.

Not considered in our calculations are extremely small values of λ, say, less than 0.2. Then, eq. (28) of ref. 21, applied to estimate the ratio $\hbar\omega_D/2k_BT^*$ lets us put $\hbar\omega_D/2k_BT^* \geq 1.5$ in (B·2) (see Appendix B) to solve eq. (B·2) numerically. Calculations are repeated with different λs which run up to very high values, looking for how strongly the ratio $2E_g(0)/k_BT^*$ changes with λ. Although approximations made to give (17)–(18) are justifiable only in the limit of weak coupling, we found that the description of the effects associated with the appearance of nonzero averages $\langle b_{\mathbf{K}}\rangle$ and $\langle b_{\mathbf{K}}^{+}\rangle$ in the system which was the main motivation in truncating the infinite chain of equations (7) are already captured qualitatively in the lowest order of λ. The variation in $2E_{\rm g}(0)/k_{\rm B}T^{*}$, i.e., the change in the slope of the curves of Fig. 4, turns out to be small when λ is larger than about 0.8. For example, $2E_{\rm g}(0)/k_{\rm B}T^{*} \simeq 3.2$ for $\lambda = 0.86$ and changes by only about 10% as λ rises increases to 3. However, $2E_{\rm g}(0)/k_{\rm B}T^{*}$ changes by less than 2% on further increasing λ and continues to rise but very slowly, saturating to a constant value of about 3.67. Thus, $2E_{\rm g}(0)/k_{\rm B}T^{*}$ increases no more than 10 to 12% as λ increases from 0.8 to very large values. As to the ratio $\hbar\omega_{\rm D}/2k_{\rm B}T^{*}$ its increase leads to a small variation in the value of 3.67 at which $2E_{\rm g}(0)/k_{\rm B}T^{*}$ saturates. In agreement with our estimations for λ made above, we focus on $\lambda = 0.86$ (dotted curve) and 1.2 (dashed curve) in Fig. 4.

Increasing λ beyond 0.86 gradually increases the slopes in Fig. 4, approaching a limiting value for $2E_{\rm g}(0)/k_{\rm B}T^{*}$ which appears to be very close to that given by the s-wave-pairing BCS ratio $2\Delta(0)/k_{\rm B}T_{\rm c} \simeq 3.53$. The unusual high λs as well of $\hbar\omega_{\rm D}/2k_{\rm B}T^{*}$ lead to slightly higher $2E_{\rm g}(0)/k_{\rm B}T^{*}$ values approaching 3.7. The solid line in Fig. 4 exhibits $2E_{\rm g}(0)/k_{\rm B}T_{\rm c}$ vs $T^{*}/T_{\rm c}$ for $\lambda = 3$. To summarize the numerical analysis, although the ratio $2E_{\rm g}(0)/k_{\rm B}T^{*}$ *does* depend on λ it varies very weakly for $\lambda \gtrapprox 0.8$ since as one increases λ from 0.86 up to very large λs the ratio $2E_{\rm g}(0)/k_{\rm B}T^{*}$ rises very slowly approaching the value 3.7. But it remains strictly between 3.2 and 3.7 for any $\lambda \geq 0.86$. This provides an alternate explanation for the experimental data collected in Fig. 5 of ref. 2 and reproduced in our Fig. 4 where $2E_{\rm g}(0)/k_{\rm B}T_{\rm c}$ is shown as a function of $T^{*}/T_{\rm c}$ for various values of λ. As seen from Fig. 4, for a given HTSC compound with a large ratio $T^{*}/T_{\rm c}$, the magnitude of $2E_{\rm g}(0)/k_{\rm B}T_{\rm c}$ is expected to be large. This qualitatively agrees with the results of refs. 38, 42, 43, 45–47 where the experimental data in $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Bi2212), La_2CuO_4 (La214) and $YBa_2Cu_3O_{7-\delta}$ (Y123) are reported. In particular, Fig. 4 explains experiments on overdoped $Bi_2Sr_2CuO_{6+\delta}$ (Bi2201)[2] where $T_{\rm c}$ was only 10 K, but with the surprisingly large $2E_{\rm g}(0)/k_{\rm B}T_{\rm c} \simeq 28$. According to Fig. 4 the ratio $T^{*}/T_{\rm c}$ appears extremely large, specifically $\simeq$7, as in (21) and is in a good agreement with ref. 2. It was shown in ref. 44 how the momentum-space symmetry of the pseudogap and superconducting gaps are the same. Furthermore, the pseudogap is not restricted to the underdoped state, but occurs also in optimally-doped and overdoped superconducting compounds such as $Bi_{2.1}Sr_{1.9}CaCu_2O_{8+\delta}$.[38,45] These observations suggest that the pseudogap and superconducting gap have a common origin, indirectly supported by the present work according to which both gaps are experimental manifestations of *the same phenomena* displayed in two distinct temperature regions, namely $T_{\rm c} \leq T \leq T^{*}$ and $T \leq T_{\rm c}$.

7. Conclusions

It was shown that an upward shift of $2\Delta(\lambda)$ between the boson and fermion spectra leads on cooling to a continuous decrease of the fermion chemical potential $\mu(\lambda, T)$ with respect to the value $E_{\rm F}$ associated with interactionless fermions at very small T. In other words, the BF binary gas mixture state develops gradually from the fermion gas as T is lowered. According to (17), (19) and (20) the single-fermion spectrum becomes gapped as the difference $E_{\rm F} - \mu(\lambda, T)$ starts differing from zero, i.e., already beginning at and below a temperature $T = T^{*} > T_{\rm c}$ so that a minimum energy $E_{\rm g}(\lambda, T)$ is required to excite single fermions from the subsystem of unpaired fermions in the BF mixture. An analytic expression for the value of the gap in the spectrum of single-fermions which evolves by bosonization is therefore derived via two-time Green functions.

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Appendix A

Write the bosonic operators $b_{\mathbf{K}}$ in terms of local scalar field operators $\phi(\mathbf{r})$ as in ref. 7, namely

$$b_{\mathbf{K}} = L^{-d/2}\int d\mathbf{r}\,\phi(\mathbf{r})\exp(-\mathrm{i}\mathbf{K}\cdot\mathbf{r}) \tag{A·1}$$

and consider the averages

$$\frac{\langle b_{\mathbf{K}}^{+}b_{\mathbf{K}}\rangle}{L^{d}} = L^{-2d}\int_{L^d}\int_{L^d} d\mathbf{r}_1\,d\mathbf{r}_2\langle\phi^{+}(\mathbf{r}_1)\phi(\mathbf{r}_2)\rangle \times\exp(\mathrm{i}\mathbf{K}\cdot[\mathbf{r}_1-\mathbf{r}_2]). \tag{A·2}$$

In the thermodynamic limit when $L^{d}\to\infty$ (with the density $\langle b_{\mathbf{K}}^{+}b_{\mathbf{K}}/L^{d}\rangle$ of CPs of total CMM K constant) the integrals over r_1 and r_2 in (A·2) accumulate mainly at largely separated points $\mathbf{r}_1$ and $\mathbf{r}_2$. Indeed, the portions of neighboring points r_1 and r_2 contributing to (A·2) are much smaller than portions due to remote r_1 and r_2. When referred to well-separated points r_1 and r_2 correlations between $\phi^{+}(r_1)$ and $\phi(r_2)$ weaken and one may write

$$\begin{aligned}\langle b_{\mathbf{K}}^{+}b_{\mathbf{K}}/L^{d}\rangle &\simeq L^{-2d}\int_{L^d} d\mathbf{r}_1\langle\phi^{+}(\mathbf{r}_1)\rangle\exp(\mathrm{i}\mathbf{K}\cdot\mathbf{r}_1)\\ &\quad\times\int_{L^d} d\mathbf{r}_2\langle\phi(\mathbf{r}_2)\rangle\exp(-\mathrm{i}\mathbf{K}\cdot\mathbf{r}_2)\\ &= \langle L^{-d/2}b_{\mathbf{K}}^{+}\rangle\langle L^{-d/2}b_{\mathbf{K}}\rangle.\end{aligned} \tag{A·3}$$

Justification of $\langle b_{\mathbf{K}}^{+}b_{\mathbf{K}}\rangle \simeq \langle b_{\mathbf{K}}\rangle\langle b_{\mathbf{K}}^{+}\rangle$ given here and used in (13) was adopted from ref. 32, p. 57 where this approximation was justified in detail for $K = 0$ whenever $L^{d}\to\infty$. Note, in the pure Bose gas considered in ref. 32, the nonzero averages $\langle b_{\mathbf{0}}\rangle$ and $\langle b_{\mathbf{0}}^{+}\rangle$ emerged after eliminating (in a manner as discussed above) the degeneracy associated with boson-number conservation-law. But because of (2) the bosons number does not conserved in a BF mixture gas. As a result, the condition $\langle b_{\mathbf{K}}\rangle = \langle b_{\mathbf{K}}^{+}\rangle = 0$ *exact for* the pure Bose system with boson-number conservation fails. Being different from zero below T^{*}, the mean values of the bosonic annihilation and creation operators appear to be related through the (A·3).

Appendix B

To see how the ratio $2E_{\rm g}(0)/k_{\rm B}T_{\rm c}$ differs in various HTSCs with different values of $T^*/T_{\rm c}$ and get Fig. 4 we introduce the new dimensionless variables

$$X \equiv \frac{T^*}{T_{\rm c}}, \quad Y \equiv \frac{\Delta(0)}{2k_{\rm B}T_{\rm c}}, \quad \text{and} \quad D \equiv \frac{\hbar\omega_{\rm D}}{2k_{\rm B}T^*} \qquad ({\rm B}\cdot 1)$$

in terms of which (27) takes the form

$$\frac{1}{\lambda^2}\left(\frac{Y}{DX}\right)^2 + \int_{-D}^{D}\frac{\tanh x}{x}\,{\rm d}x - \ln\left[1 + 2\left(\frac{DX}{Y}\right)^2 + 2\frac{DX}{Y}\sqrt{1+\left(\frac{DX}{Y}\right)^2}\right] = 0. \qquad ({\rm B}\cdot 2)$$

where both X and Y are functions of λ and $\hbar\omega_{\rm D}$. However, as discussed in Fig. 2, for fixed λ the parameter D in (B·2) does not depend on $\hbar\omega_{\rm D}$. If one fixes the value of λ then the ratio D in (B·2) may be considered constant, the actual value of which must be defined with a choice of λ. That is, *for fixed* λ (27) written in terms of X, Y, and D allows us to consider it as a relation between *two* independent variables, i.e., between $2\Delta(0)/k_{\rm B}T_{\rm c}$ and $T^*/T_{\rm c}$ which may be solved, at least numerically, to find Y satisfying (B·2) for a given X.

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