Physica E 44 (2011) 394–399

Contents lists available at SciVerse ScienceDirect

Physica E

journal homepage: www.elsevier.com/locate/physe

Low-dimensional Fermi and Bose gases

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ARTICLE INFO

Article history:
Received 7 April 2011
Received in revised form
21 August 2011
Accepted 7 September 2011
Available online 10 September 2011

ABSTRACT

In the ideal Fermi gas a physical interpretation can be given to a curious "hump" that develops in the chemical potential $\mu(T)$ as a function of absolute temperature T for any spatial dimensionality $d<2$, integer or not. This contrasts with the more familiar monotonic decrease for $d \geq 2$. The hump height increases without limit as d decreases toward zero where $\mu=+\infty$. This positive divergence at $d \to 0$ is argued to be a clear manifestation of the Pauli Exclusion Principle *in configuration space*, whereby two spinless fermions cannot sit on top of each other. The observed hump is thus an obvious precursor of this manifestation, otherwise well understood, say, in the 1s level of the H atom. The ideal Bose gas for $d<2$ is also reexamined and found impossible to be confined at all in $d \to 0$ as it exhibits the opposite divergence $\mu=-\infty$ there. Both divergences are seen to follow from the Heisenberg Uncertainty Principle.

1. Introduction

The ideal Fermi gas (IFG) possesses a peculiar nonmonotonic "anomaly" for all dimensions $d<2$ [1–4]. The anomaly seems to be intimately connected with the Pauli Exclusion Principle (PEP) [5]. The PEP plays a well-known role in constructing the Periodic Table; in ionic and covalent bonding in molecules and solids; in metals, insulators, semiconductors, superconductors; in nuclear shell structure and binding energy including neutron stability when in a nucleus, to name just a few applications. The PEP has been shown [6,7] to be responsible even for the fact that ordinary bulk matter is stable and occupies volume. This is in addition to the easily established remarkable fact that already the simplest atom H is 99.9999999999999% empty space [8].

On the other hand, it is known that the nonrelativistic ideal Bose gas (IBG) undergoes a Bose–Einstein condensation (BEC) *only* for any dimensionality $d>2$, integer or not, with a cusp singularity in the temperature-dependent heat capacity for $2<d<4$ and a finite jump discontinuity for all $d>4$, see, e.g., Refs. [9–11]. Since its theoretical prediction by Einstein in 1925 based on the work in 1924 by Bose on photons, and after languishing for seven decades as a mere academic exercise in textbooks, BEC has been observed in the laboratory in laser-cooled, magnetically trapped ultra-cold bosonic atomic clouds of $^{87}_{37}$Rb atoms [12], $^{7}_{3}$Li [13], $^{23}_{11}$Na [14], $^{1}_{1}$H [15], $^{85}_{37}$Rb [16], $^{4}_{2}$He [17], $^{41}_{19}$K [18], $^{133}_{55}$Cs [19], $^{174}_{70}$Yb [20] and $^{52}_{24}$Cr [21]. Here, the upper and lower prefixes are the nuclear mass number (of nucleons in the atomic nucleus) and proton numbers, respectively. BEC in gases of excitons [22] and of magnons [23–25] has also been reported. It has even been observed in lower dimensions: Görlitz et al. [26] report BEC of $^{23}_{11}$Na atoms in 1D or 2D; Schreck et al. [27] observe it with $^{7}_{3}$Li atoms in 1D; and Burger et al. [28] study the phase transition in a cloud of $^{87}_{37}$Rb atoms in quasi-2D.

The discovery of the quasi-2D superconductors such as the cuprates [29–31] or the quasi-1D superconductors like the organo-metallic (or Bechgaard) salts [32–34] and carbon nanotubes [35], have further motivated studying low-dimensional quantum gases, with or without interatomic interactions. Cuprate superconductors are well-known to have a laminar structure along directions, say a and b which are perpendicular to direction c. Resistivity ρ anisotropies ρ_c/ρ_{ab} where ab denotes the CuO or BaO or SrO planes, as the case may be, can be as high as 10^5 in $Bi_{2+x}Sr_{2-y}CuO_{6+\delta}$ [36] if not higher, even though only about 10^2 in $YBa_2Cu_3O_{7-\delta}$ (*YBCO*). In the Drude 1900 resistivity model [37, p. 7] $\rho=m/ne^2\tau$ for current carriers of charge e, effective mass m and number density n, while τ is some average time between collisions. Thus, if $\rho_c/\rho_{ab}=m_c/m_{ab}$ is ∞ one has a precisely 2D situation; if it is 1 we have the perfectly isotropic 3D case. Hence, the large but finite ratio 10^5 observed implies $(2+\varepsilon)D$ or "quasi-2D" behavior, with ε small. Even for *YBCO* the value $d \simeq 2.03$ has been extracted independently by two groups [38,39] and suggested to be more realistic than $d=2$, since $d \simeq 2.03$ reflects inter-CuO (or BaO or SrO as the case may be) -layer couplings.

In contrast to bosons with singular behavior *above*2D (the well-known BEC), fermions exhibit what is arguably nontrivial

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doi:10.1016/j.physe.2011.09.003

anomalous behavior *below*2D [1–4]. Low-dimensional Fermi systems have found a host of practical electronic-device applications [40–43] with quantum "wells" (2D), "wires" (1D) and "dots" [44] (0D).

Noninteger or fractal dimensions are used, e.g., to compare the complexity fractals of two curves or two surfaces [45]. In materials science a noninteger dimension is directly related to "roughness" [46] and leads to basic applications in stereology [47], powder technology [48,49], geology [50], metallurgy [51], computer graphics [52–54], and so on. Since "fractal" means "broken" or "fractured" noninteger dimensions falls within this definition, and can otherwise be associated with an irregular measurement between integer dimensions. For instance, it is known that Brownian motion is a type of random walk [55] which is fractal. Fractals occur in a wide range of phenomena, from river levels and landscape topography to computer network traffic and stock-market indicators [56]. Indeed, Mandelbrot [56, p. 85] cites an empirical fractal dimension $d=1.23$ for the distribution of galaxies in the observable universe. Thus, a fractal geometry for Bose and Fermi gases at low dimensions is possibly applicable.

It is well-known from textbooks that the IFG chemical potential $\mu(T)$ as function of absolute temperature T monotonically *decreases* with T for both $d=2$ and 3 from its $T=0$ value, the Fermi energy $E_F \propto \hbar^2 n^{2/d}/m$ (with n and m the fermion particle density and mass, respectively). But for 1D a peculiar a rise from its $T=0$ value of E_F appears in $\mu(T)$ at low temperatures before decreasing to the classical limit that diverges negatively as $-T\ln T$. A nonmonotonic "humped" shape can then be surmised from a *rise* of $\mu(T)$ for $d=1$ as reported [2] in a figure without further comment. They determined this peculiar rise from the Sommerfeld low-T series expansion [37, pp. 45ff.] which for $d \neq 2$ begins with [1–4]

$$\mu(T)/E_F \underset{T/T_F \to 0}{\longrightarrow} 1-(d-2)(\pi^2/12)(T/T_F)^2 + O(T^4), \tag{1}$$

where $T_F \equiv E_F/k_B$ is the Fermi temperature. Thus, the first correction to unity is *positive* for all $d<2$. For $d=2$ the density of states is energy-independent, which allow deriving the *exact* explicit expression

$$\mu(T)/E_F = T/T_F \ln[\exp(T_F/T)-1] \underset{T\to 0}{\longrightarrow} 1. \tag{2}$$

In contrast with all $d \neq 2$, this is *not* expandable in powers of T/T_F as $\exp(T_F/T)$ evidently has an *essential singularity* at $T=0$.

A well-developed hump behavior in the IFG $\mu(T)$ for any $d<2$ has indeed been found numerically and reported in greater detail in Refs. [3,4]. The hump height rises without limit as $d \to 0$. This same behavior then showed up in the heat capacity and is reminiscent of the Schottky effect [2,57,58] in paramagnetic salts at low temperatures. Though not singular behavior as such in the IFG, it still remained an intriguing *anomaly*. May [9] has investigated the peculiar role of the specific value $d=2$ dividing behaviors in the IFG and the IBG, showing how their specific heats are *identical* for all T at precisely $d=2$ itself. This holds for *quadratically* dispersive particles.

Even *fermionic* atomic gases like ${}^{40}_{19}\mathrm{K}$ [59] and ${}^{6}_{3}\mathrm{Li}$ [60] exhibit what seems to be a BEC, in what some consider the "sixth" state of matter—if BEC is the "fifth". The sixth state results when some of the fermions presumably Cooper-pairing [61] into bosons undergo BEC. The IFG has been thoroughly treated in general by many authors [1–4,58] and detailed studies of the quantum behavior in any dimension at sufficiently low temperatures in these systems has also gained interest as possible precursors of the presumed paired-fermion condensate at lower temperatures, in hopes that it might shed light on the phenomenon of superconductivity supposedly presaged by the pairing of *some* electrons (or holes) to form a boson-fermion *binary* mixture with a subsequent BEC of the bosonic Cooper-pair subsystem (a brief survey for *ternary* mixtures including hole pairs is in Ref. [62]; a more extensive treatment is Ref. [63]).

In Section 2 we revisit the polylogarithm function by reexamining the IFG of quadratically dispersive fermions in any space dimension $d \geq 0$, integer or not. The hump for all $d<2$ is interpreted as a precursor of the limitless value taken by the maximum value of $\mu(T)$ as $d \to 0$ is approached, which in turn is argued to be a manifestation in configuration space of the PEP for spinless fermions. In Section 3 the Heisenberg Uncertainly Principle (HUP) is invoked to shed light on both fermion and boson divergences; in Section 4 the boson case is discussed and Section 5 presents conclusions.

2. Ideal Fermi gas in $d \leq 2$

Consider a d-dimensional noninteracting gas of N identical bosons or N identical fermions of mass m moving freely, i.e., with a quadratic dispersion relation as implied by the Hamiltonian $H=\sum_{i=1}^{d} p_i^2/2m$. This gives rise to eigenvalues $\varepsilon_{\mathbf{k}} = \hbar^2 k^2/2m$ and ultimately to the boson or fermion number equation

$$N = \sum_{\mathbf{k}} [z^{-1}\exp(\beta\varepsilon_{\mathbf{k}})+a]^{-1}, \tag{3}$$

where $\beta \equiv 1/k_B T$, $\mu(T)$ is the chemical potential, and $z \equiv \exp\beta\mu(T)$ is the real gas fugacity. For bosons $a=-1$ while for fermions $a=1$ and for "boltzons" $a=0$. In a cubical "box" of size L of d dimensions where L^d is the system volume, the summation over $\mathbf{k}$ can be replaced by an integral, namely

$$\sum_{\mathbf{k}} \longrightarrow (L/2\pi)^d \int d^d k. \tag{4}$$

Thus, in a volume L^d in any dimension d and in the continuum limit Eq. (3) becomes

$$N = (L/2\pi)^d [2\pi^{d/2}/\Gamma(d/2)] \int \frac{dk k^{d-1}}{z^{-1}\exp(\beta\varepsilon_{\mathbf{k}})+a}. \tag{5}$$

Here the volume of a hypersphere of radius R in $d \geq 0$ dimensions [58]

$$V_d(R) = \pi^{d/2} R^d/\Gamma(1+d/2) \quad \textbf{note}: \; V_0(R) \equiv 1 \tag{6}$$

was used. Integrating over $x \equiv \beta\varepsilon_{\mathbf{k}}$ instead of over k in Eq. (5) introduces the expression

$$\frac{1}{\Gamma(\sigma)} \int_0^\infty dx \frac{x^{\sigma-1}}{z^{-1}\exp x + a} = -\frac{1}{a} \sum_{l=1}^{\infty} \frac{(-az)^l}{l^\sigma} \equiv -aLi_\sigma(-az) \quad |z|<1. \tag{7}$$

Here $Li_\sigma(t) = \sum_{l=1}^{\infty} t^l/l^\sigma$ with ($|t|<1$) is the polylogarithm function; it is designated as $PolyLog[\sigma,t]$ in Ref. [64]. This function has important applications in both physics and mathematics such as number theory, representation theory of infinite dimensional algebras, exact soluble models, conformal theories, d-dimensional analysis, etc. [65–72]. The limitation $|z|<1$ in convergence arises from the small-z binomial expansion of the integrand on the lhs of Eq. (7), which is then integrated term by term to get the infinite summation. For fermions we shall need to go beyond this unit circle of convergence in the z-plane, in fact the case $z \to \infty$ will be required. For $a=-1$ Eq. (7) is the Bose integral $g_\sigma(z)$ which for $z=1$ and $\sigma \geq 1$ becomes the Riemann Zeta function $\zeta(\sigma)$ of order σ. For $a=1$ Eq. (7) is the Fermi integral $f_\sigma(z)$. Both integrals are extensively discussed in Ref. [58, Appendix].

Introducing the standard thermal wavelength

$$\lambda \equiv h/\sqrt{2\pi m k_B T} \tag{8}$$

where h is Planck's constant, from Eqs. (5) and (7) with $\sigma = d/2$ one has the reduced (i.e., dimensionless) number density for spinless fermions

$$n\lambda^d = \frac{1}{\Gamma(d/2)} \int_0^\infty dx \frac{x^{d/2-1}}{\exp(x-\alpha)+1} \equiv I_{d/2}(\alpha). \tag{9}$$

Here $n \equiv N/L^d$ and $\alpha(T) \equiv \beta\mu(T) \equiv \ln z$. Letting $\exp(\alpha - x) \equiv y$ in Eq. (9) and calling $d/2-1 \equiv m$ one gets the integral representation [73]

$$I_{m+1}(z) = \frac{1}{\Gamma(m+1)} \int_0^z \frac{dy}{y+1}(\alpha - \ln y)^m, \quad 0 < z < \infty, \tag{10}$$

which is now valid for *all* nonnegative z. It can in fact be shown (Ref. [73], Eq. (10)) that the integral defined in Eq. (9) is precisely

$$I_m(z) \equiv -Li_m(-z) \tag{11}$$

where the polylog function $Li_m(z)$ can be defined as

$$Li_m(z) = \int_0^z \frac{dt}{t} Li_{m-1}(t), \quad 0 < z < \infty \tag{12}$$

or, alternately by the recurrence relation

$$z\frac{\partial}{\partial z} Li_m(z) = Li_{m-1}(z). \tag{13}$$

This in turn immediately implies that

$$Li_1(z) = -\ln(1-z) \quad \text{and} \quad Li_0(z) = z/(1-z), \tag{14}$$

where the latter expression is to be used in the $d \to 0$ for IFG case to be discussed. From Eqs. (9) and (11) the number density $n \equiv N/L^d$ of spinless fermions is thus given by

$$n\lambda^d = I_{d/2}(z) = -Li_{d/2}(-z). \tag{15}$$

As mentioned, one sees an anomalous behavior in the chemical potential of an IFG [4] as $d \to 0$. There is another anomaly with number density as $d \to 0$. From Eq. (15) as $d \to 0$ and the rhs in Eq. (14), the IFG number density becomes

$$n_{d\to 0} = \frac{z}{1+z}. \tag{16}$$

As n must be T-independent so must z and the only possible *unique* solution to Eq. (16) is $z \equiv \exp \beta\mu(T) \to +\infty$, implying that $n_{d\to 0} \equiv N/L^{d\to 0} \to 1$. Hence, in precisely null dimension (0D) only *one* spinless Fermi particle (or two for spin $\frac{1}{2}$, etc.) can be accommodated. It is a thermodynamic statement of the PEP in *coordinate space* analogous to that in momentum space, viz., the 1s state of the hydrogen atom. From Eq. (16) one obtains as we exhibit in Fig. 3 $\beta\mu = \ln[n_{d\to 0}/(1-n_{d\to 0})]$ whereas the thermodynamic limit (TL) holds every step of the way (see Appendix).

The dimensionless internal energy U of the IFG in terms of polylogs [74] is given by

$$2\beta U/dN = \frac{Li_{d/2+1}(-z)}{Li_{d/2}(-z)}. \tag{17}$$

The specific heat at constant volume $C_V(T) = (\partial U/\partial T)_{L^d}$ is then given by

$$\frac{2C_V(T,L^d)}{dNk_B} = (d/2+1)\frac{Li_{d/2+1}(-z)}{Li_{d/2}(-z)} - (d/2)\frac{Li_{d/2}(-z)}{Li_{d/2-1}(-z)}, \tag{18}$$

where Eq. (13) was used. A plot of $C_V(T)$ for several values of d is shown in the rhs inset of Fig. 2 as function of T/T_F.

Numerical studies [4] of the low-temperature Fermi chemical potential for $0 < d < 2$, Fig. 1, show that the hump observed in 1D [2] is not an isolated or accidental feature. It seems to be foreshadowing a deeper physical principle. Its full significance can only be extracted when d is taken to zero, or 0D. In an effort to understand the confineability of particles, a study of physical properties in 0D was made [75, p. 947] by analytically continuing d to any dimension [76], integer or not. It was possible to do so since the number density is expressed in the transcendental function of z and d, namely $Li_{d2}(z)$. The 0D behavior Eq. (16) is the limit of the results obtained by the numerical studies just mentioned. As the density $n_{d\to 0}$ it must be T-independent requiring that $z \equiv \exp \beta\mu(T) \to +\infty$ at any T, resulting in $n_{d\to 0} \to 1$ for spinless fermions. The above result [75] is obtained by taking $d \to 0$ first and $z \to +\infty$ second, where the second limit is physically driven. Namely, if $d \to 0$, z becomes strongly d-dependent such that $z \to +\infty$. That $z \equiv \exp \beta\mu(T) \to +\infty$ implies that this occurs at *any* T. Thus in 0D, T is an irrelevant quantity. As noted in Ref. [73] there are no classical solutions in 0D and only the ground state is important in Eq. (4). That at 0D, $\mu \to +\infty$ is precisely the limit that the numerical results of Fig. 1 suggest for any d, integer or not. We recall that μ is the energy required to bring another Fermi particle into the system volume containing other Fermi particles. At $T=0$ this energy is just the Fermi energy E_F. In 0D it should not be physically possible owing to the PEP. This impossibility is manifested with $\mu \to +\infty$ as an impenetrable barrier to other Fermi particles. From Eq. (17) as $d \to 0$ one has $2\beta U/dN = (1+z)ln(1+z)/z$. Hence, since $z \to +\infty$ implies that the dimensionless internal energy diverges, $2\beta U/dN \to +\infty$.

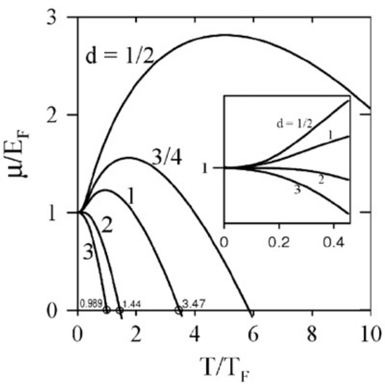

Fig. 1. Chemical potential $\mu(T)$ (in units of Fermi energy E_F) of an IFG in $d = 3, 2, 1, \frac{3}{4}$ and $\frac{1}{2}$ spatial dimensions as function of absolute temperature T (in units of Fermi temperature $T_F \equiv E_F/k_B$). The monotonically decreasing curves for $d=2$ and 3 are the familiar textbook results. They turn negative at $T/T_F \simeq 1.44$ and 0.989, respectively. Inset illustrates *rise* of $\mu(T)$ with T for all $d < 2$, as opposed to its well-known monotonic decrease for all $d \geq 2$.

For *linearly* (as opposed to quadratically) dispersive quantum particles the dividing d-value associated with the IFG and the IBG, i.e., above which BEC occurs for bosons and below which a nonmonotonic $\mu(T)$ for fermions appears, is not $d=2$ but rather $d=1$, as already also inferred in Ref. [77]. Pathria [78] has explicitly associated the dividing d-value with the exponent $s > 0$ in the particle dispersion relation $\varepsilon_{\mathbf{k}} \propto k^s$ where k is again the momentum wavenumber. Examples of $s=1$ are: (i) nonrelativistic bosonic Cooper pairs [79] in a Fermi sea in leading order in k or (ii) ultrarelativistic bosons [80] *in vacuo*. Cooper pairs [61], in contrast to BCS pairs [81], are bosonic [82,83] in that their energy depends only on their center-of-mass momenta but *not* also on their relative momenta as do BCS pairs; thus Cooper pairs obey Bose statistics. In general, bosons suffer a BEC *only* for all $d > s$ below a critical temperature $T_c \propto \hbar^2 n^{s/d}/k_B m$ [84]. Only just recently did a *non*relativistic example of $s=1$ *fermions* surface: the so-called Dirac electrons in graphene which is apparently the first truly 2D actual material (two excellent reviews are Refs. [85,86]).

3. Uncertainty principle and μ-divergences

As $d\to 0$ the volume $L^d \to 1$. Since there is then no length scale, the volume in null dimension is unity, nevertheless TL is still valid (see Appendix). This suggests a point or a dot. A particle in it can have no indeterminacy Δx in position so that as $d\to 0$, $\Delta x \to \pm\ 0$. Thus, by the Heisenberg Uncertainty Principle the indeterminacy in momentum Δp must approach $\pm\infty$ [74]. Also, the time indeterminacy $\Delta t \to \pm\ 0$ is accompanied by an energy indeterminacy $\Delta\varepsilon \to \pm\ \infty$.

This holds for the IFG at $d\to 0$ while for the IBG there occurs another indeterminacy Δt from the HUP. For the IBG one has

$$2\beta U/dN = Li_{d/2+1}(z)/Li_{d/2}(z). \tag{19}$$

Thus, as $d\to 0$ the dimensionless internal energy $2\beta U/dN \to$ constant because $z\to 0 \Rightarrow \mu \to -\infty$. Then $2\beta U/dN \to 1$, while $z\to 1 \Rightarrow \mu\to 0$ so that $2\beta U/dN \to 0$ for spinless bosons. This is depicted in Fig. 4. A finite internal energy U implies from the HUP that $\Delta t\to\infty$. This implies that $\Delta p \equiv m(\Delta x/\Delta t)\to 0$ and again from the HUP that $\Delta x\to\infty$. Therefore, bosons cannot be confined at all in null dimension.

Thus the internal energy U indeterminacy $\Delta U \equiv \sum \Delta\varepsilon n(\varepsilon)$, where $n(\varepsilon)$ is defined in Eq. (3), approaches $+\infty$ for the IFG while for the IBG is finite. By the Second Law of Thermodynamics $\mu = [\Delta U/\Delta N]_{S,L^d}$ where S is the system entropy. Fixing S such that near the ground state it nearly vanishes (by the Third Law), one concludes that $\mu \to \pm\infty$ as $d\to 0$. In general, a Fermi gas near the ground state has $\mu > 0$, while a Bose gas must have $\mu \le 0$ for all states to ensure that the summands $n(\varepsilon_{\mathbf{k}})$ in Eq. (3) are never negative. The HUP thus demands that as $d\to 0$, $\mu\to+\infty$ for a Fermi gas and $\mu\to-\infty$ for a Bose gas. Finally, from Eq. (3) $n(\varepsilon)\to 1$ for fermions and $n(\varepsilon)\to 0$ for bosons. We now illustrate the case of bosons.

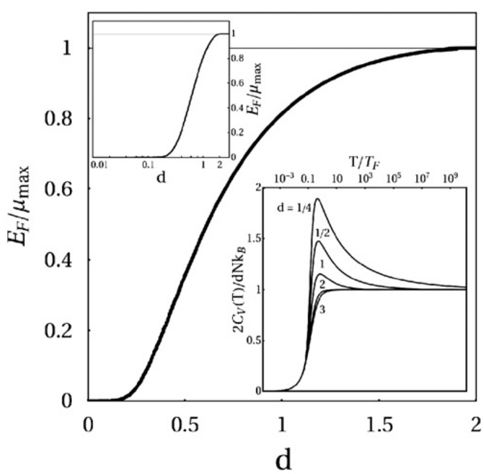

Fig. 2. Inverse of the*maximum* value of the $\mu(T)/E_F$ for an IFG for all $d<2$ displaying how $\mu(T)/E_F$ must diverge positively as $d\to 0$. Left inset is a semilog plot. As expected, maximum value of $\mu(T)$ is just E_F for all $d\ge 2$ (thin horizontal line at top of both figures). Right inset illustrates on a semilog plot humped behavior for all $d<2$ of specific heat $C_V(T)$ which for $d\ge 2$ is monotonic increasing.

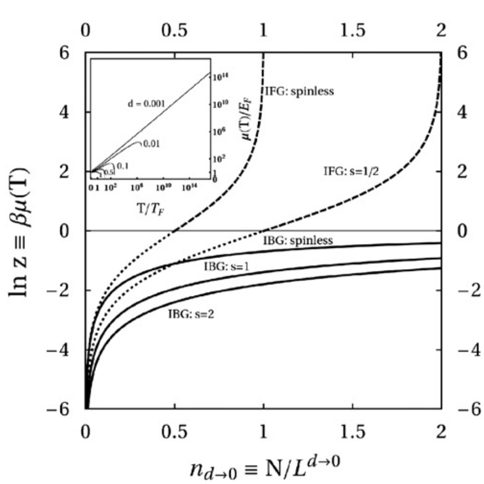

Fig. 3. For the IFG we plot $\ln(z) \equiv \beta\mu(T)$ vs. number density $n_{d\to 0}$, from Eq. (16) arbitrarily near null dimension (long-dashed curves). For spinless fermions $\mu\to+\infty$ as $n_{d\to 0}\to 1$ and $n_{d\to 0}\to 2$ for spin $s=\frac{1}{2}$ and the TL is preserved, i.e., $n_{d\to 0}$ is finite. Short-dashed extensions below thin horizontal line at $\ln(z)\equiv\beta\mu(T)=0$ are nonphysical as they violate positivity $\mu>0$ for the IFG as $d\to 0$. This positivity is clearly evident in Inset. For the IBG (solid-line curves) from Eq. (23) at $d\to 0$ as $z\to 0 \Rightarrow \mu\to-\infty$ so that the number density converges $n_{d\to 0}\to 0$, i.e., the TL is preserved, while $z\to 1 \Rightarrow \mu\to 0$ so that number density diverges $n_{d\to 0}\to\infty$ and the TL is not valid in null-d.

4. Ideal Bose gas in $d\le 2$

For bosons ($a=-1$) in Eq. (3) so that instead of Eq. (15) one has [75]

$$n\lambda^d = Li_{d/2}(z) \tag{20}$$

or in general that $n\lambda^d = -a^{-1}Li_{d/2}(-az)$. When $z\equiv \exp\beta\mu = \exp\beta_c 0 = 1$. If Eq. (8) for $T=T_c$ is λ_c, in $3D$ this implies that $n\lambda_c^3 = Li_{3/2}(1) \equiv \zeta(3/2)$ the Riemann zeta function, which in turn immediately leads to

$$T_c = 2\pi\hbar^2 n^{2/3}/m\zeta(3/2)^{2/3} \tag{21}$$

or the familiar BEC critical temperature.

Without loss of generality, we scale the *boson* number density n in Eq. (20) with the number density of *fermions* in an IFG in any dimension $n_F \equiv (k_F^2/4\pi)^{d/2}/\Gamma(d/2+1)$ where $k_B T_F \equiv E_F \equiv \hbar^2 k_F^2/2m$ is the Fermi energy of an equivalent system of fermions. Thus

$$n/n_F = \Gamma(d/2+1)(T/T_F)^{d/2} Li_{d/2}(z). \tag{22}$$

The boson chemical potential $\mu(T)$ for any dimension $d>0$ follows from Eq. (22). In Fig. 5 we plot the IBG chemical potential $\mu(T)$ in units of the E_F for several dimensionalities $d\le 2$ down to $d=0.001$ as a function of T in units of T_F and for $n/n_F=1$. In the Inset we fixed the dimension at $d=0.1$ and display results for the five distinct values of n/n_F shown.

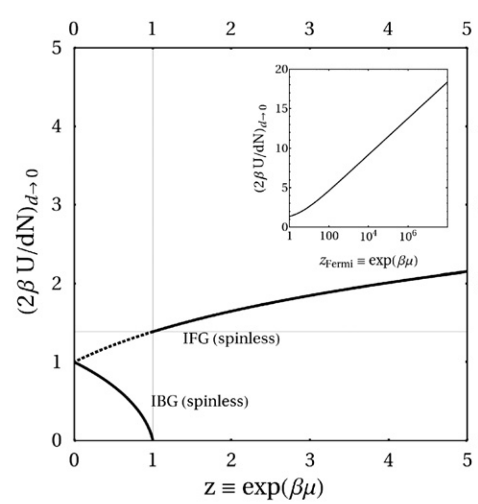

Fig. 4. Dimensionless internal energy $2\beta U/dN$ for IBG and IFG spinless systems as $d\to 0$. For the Fermi gas as $z\to\infty \Rightarrow 2\beta U/dN\to\infty$, this is shown in Inset, while $z\to 1 \Rightarrow 2\beta U/dN$ is finite. The short-dashed curve violates the positivity of chemical potential $\mu>0$. For Bose gas when $z\to 1 \Rightarrow 2\beta U/dN\to 0$, wich is a natural limit for bosons and as $z\to 0 \Rightarrow 2\beta U/dN\to 1$.

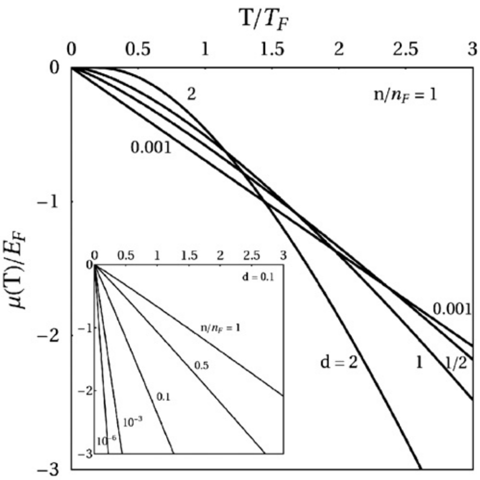

Fig. 5. Quadratically dispersive-boson chemical potential $\mu(T)/E_F$ for various dimensionalities $d \le 2$ vs T/T_F for $n/n_F=1$. Inset shows for d=0.1 several values of n/n_F.

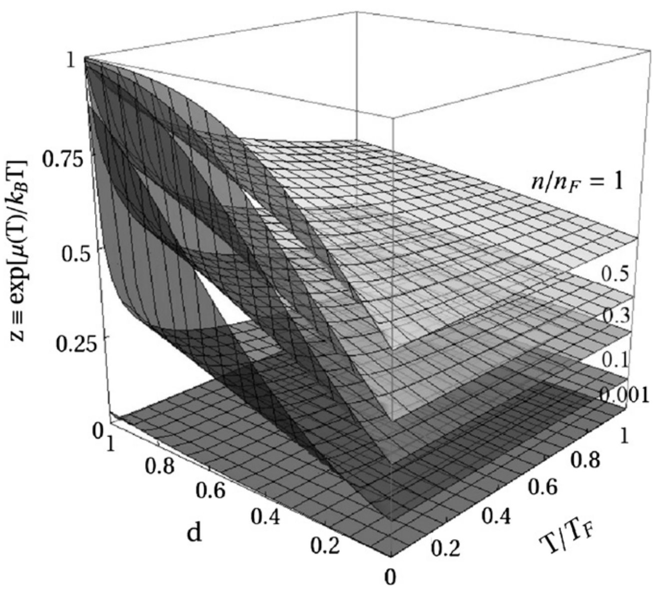

Fig. 6. Boson fugacity $z \equiv \exp\beta\mu(T)$ shown for various dimensionless number densities n/n_F from 1 down to 0.001 vs T in units of T_F. For dimensionalities $0<d<1$, $z\to 0$ and thus $\mu\to-\infty$ for all T only *provided* that $n/n_F\to 0$ for all T.

For $d\to 0$ (if d=0 precisely see Appendix A) and $a=-1$ one has from Eqs. (14) and (22) that

$$\left(\frac{n}{n_F}\right)_{d\to 0} = Li_0(z) = \frac{z}{1-z}. \tag{23}$$

Since n/n_F is a constant independent of T, so is $z \equiv \exp\beta\mu(T)$. This requires that $\beta\mu(T)$ be constant. As in the IFG case Eq. (16) it would seem that one might assign *any* arbitrary value to n/n_F which would imply that Eq. (23) possesses infinitely many solutions. However, the only true *unique* solution to Eq. (23) is with n=0 which implies that z=0 and consequently that $\mu(T)\to-\infty$ whereas the TL is still valid (see Appendix). This is shown in Fig. 3 from Eq. (23) since $\beta\mu = \ln[(n/n_F)_{d\to 0}/[1+(n/n_F)_{d\to 0}]]$. This in turn is consistent with the HUP as argued in the preceding section. This uniqueness is illustrated in Fig. 6 where one observes that $z\to 0$ for any T *only* when $n/n_F\to 0$ where temperature has no meaning. Note that even for n/n_F as small as 0.001, z continues to decrease at T=0 as $d\to 0$. All this means that bosons cannot be confined at all in null dimension since the chemical potential there is $-\infty$ and an infinitely large negative chemical potential indicates nonconfineability of the particles.

5. Conclusions

The "hump" in the low-T chemical potential $\mu(T)$ in the ideal Fermi gas (IFG) that appears for all $d<2$, grows higher as d decreases and diverges positively at $d\to 0$, i.e., $\mu\to+\infty$. The hump thus portends the existence of an infinite-potential barrier at 0D whereby there may be one and only one spinless fermion but no more. The thermodynamic limit holds always. The single spinless fermion can be seen to constitute a definition of an "ideal quantum dot" and is a manifestation of the Pauli Exclusion Principle (PEP) in configuration space.

For the ideal Bose gas its specific heat has either a cusp or a jump singularity for all $d>2$, this being a sign that it undergoes the well-known Bose–Einstein condensation. But in 0D, $\mu(T)$ has only one solution, namely $\mu\to-\infty$ which forces bosons over the edges of a potential-energy chasm that prevents ideal bosons from being confined at all in null dimension.

Thermodynamics implies that both divergences, $\mu=\pm\infty$ are a manifestation of the Heisenberg Uncertainty Principle. The additional anomaly of ideal quantum gases at *precisely* d=0 arises because the polylogarithm function as a complex function of d has a pole at precisely d=0 which is an essential singularity [75]. This was already encountered in Eq. (2) as a function of T and prevented the existence of a Sommerfeld low-T series expansion for d=2 that was otherwise possible for all $d\neq 2$.

Acknowledgments

MdeLl thanks M.H. Lee for numerous discussions and suggestions as well as for a critical reading of the MS. He also thanks A.A. Valladares for discussions. MG acknowledges partial support from Grant PE102806 (Mexico), IC from CONACyT (Mexico) Grant 103847, and MdeLl from UNAM-DGAPA-PAPIIT (México) Grant IN106908.

Appendix A. Thermodynamic limit for $d\to 0$

The thermodynamic limit (TL) corresponds to number of particles $N\to\infty$ and system volume $V\to\infty$ while the number density $n\equiv N/V$ remains constant. Since $V=L^d$ we can fix d and let $L\to\infty$. On the other hand from Eq. (5) for the IBG and the IFG ($a=\mp 1$, respectively), we have

$$N = -a\lambda^{-d}L^d Li_{d/2}(-az), \tag{A.1}$$

which when evaluated at $d\to 0$ becomes Eqs. (16) and (23). One can thus take the TL in null-d as follows. For the IFG one has from Eq. (A.1) with $d\to 0$ that $N=z(z+1)^{-1}$ so that (i) $N\to\infty \Rightarrow z\to\infty$ this implies that the chemical potential $\mu\to\infty$ at $d\to 0$; this is shown in Fig. 3 where $n_{d\to 0}\to 1$ for a spinless fermions and $n_{d\to 0}\to 2$ for a fermions with spin $s=\frac{1}{2}$. Thus the TL is preserved, i.e., $n_{d\to 0}$ is finite. Or (ii) if $N\to\infty \Rightarrow (1+z)\to 0$ then $z\to -1 \Rightarrow \mu<0$ but this violates positivity $\mu>0$ for the IFG (see dotted curves violating positivity in Fig. 3).

For the IBG from Eq. (A.1) at $d\to 0$ one gets $N=z/(1-z)$, since $z\equiv\exp[\beta\mu]$ and taking natural logarithms on both sides leads to $\beta\mu=\ln[N/(1+N)]$. In the TL (i) if $(1+N)\to\infty$ then $\mu\to-\infty$ so then number density converges, $n_{d\to 0}\to 0$. This is illustrated in Fig. 3 (solid lines for spinless bosons and bosons with s=1, s=2). Or (ii) $N\to\infty$ implies $\mu\to 0$ so that $n_{d\to 0}\to\infty$ and $z\to 1$ meaning that the TL does not hold since number density diverges. Hence only the two instances of (i) are valid for either IBG or IFG systems to determine the number density $n_{d\to 0}$ at $d\to 0$ and both preserve the TL.

The dimensionless internal energy U for IBG and IFG systems is (19) as $d \to 0$ one has the limit $(2\beta U/dN)_{d\to 0} = \ln(1+az)(1+az)/z$ where as before $a = \mp 1$ for Bose and Fermi, respectively. The specific heat C_V for the quantum ideal gases is $(\partial U/\partial T)_{L^d}$ if one approaches null dimension one has the same limit for U, i.e., for the Fermi gas: (i) $z \to 0 \Rightarrow \mu \to \infty$ and then $2C_V/dNk \to \infty$; or (ii) $z \to 1 \Rightarrow \mu \to 0$ implying that $2C_V/dNk \to const$. For the Bose gas: (i) $z \to 0 \Rightarrow \mu \to -\infty$ and then $2C_V/dNk \to 0$; or (ii) $z \to 1 \Rightarrow \mu \to 0$ and then $2C_V/dNk \to const$. When $d \to 0$ the specific heat diverges for fermions since only one particle can be confined in null-d . But it converges for the Bose gas and is equal to zero since no bosons can be confined in null dimension.

In the grand canonical ensemble (GCE) one takes a volume V with N particles. As $d \to 0$, $V \to 1$ and the "cubical box" of size L and d dimensions shrinks in size even as $L \to \infty$ while $d \to 0$. As one approaches $d=0$ only one fermion can be accommodated in this box. The length L of each cell in the GCE does not shrink but one now takes N single-cells (NSC) each of volume $V=1$ with just one fermion each so that $NSC_{fermions} \to \infty$ so that the TL is preserved. In the Bose gas, on the other hand, one cannot confine any bosons as $d \to 0$ such that every single-cell each with $V=1$ of the GCE is empty and $N_{bosons} \to \infty$, so that the TL is valid.

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