



## Study of stability of relativistic ideal Bose–Einstein condensates

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### ABSTRACT

A relativistic complex scalar boson field at finite temperature  $T$  is examined below its critical Bose–Einstein condensation temperature. It is shown that at the same  $T$  the state with antibosons has higher entropy, lower Helmholtz free energy and higher pressure than the state without antibosons – but the same Gibbs free energy as it should. This implies that the configuration without antibosons is metastable. Results are generalized for arbitrary  $d$  spatial dimensions.

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### 1. Introduction

In early works [1–3] on the relativistic ideal boson gas (RIBG) explicit Bose–Einstein condensation (BEC) critical transition temperature  $T_c$ -formulae were derived for both the nonrelativistic and ultrarelativistic limits and specific-heat anomalies at  $T_c$  were studied. In addition, Refs. [2,3] considered all space dimensions  $d > 0$  and delved into the relation between  $d$  and various critical exponents. At sufficiently high temperatures, however, boson–antiboson pair production becomes appreciable and this was *not* accounted for. The first reports to include *both* bosons and antibosons appear to be Refs. [4,5] where high-temperature expansions for the various thermodynamic functions (pressure, particle-number density, entropy, specific heats, etc.) were derived. Extensive numerical work in  $d$  dimensions that does not rely on such high-temperature expansions was reported in Refs. [6,7]. In the elegant treatment of Ref. [8] with inverse Mellin transforms the specific heat anomaly of the RIBG at its BEC  $T_c$  was found to be washed out when pair production was included. The relationship between the BEC of the RIBG and spontaneous symmetry breaking was explored in Refs. [5,9]; see also the rather complete Ref. [10], esp. §2.4.

BECs are also of interest in cosmological and astrophysical contexts. In fact, increasing attention has recently been paid cosmological models that describe dark matter (DM) as a condensate phase of some scalar boson field [11–24]. Such models are competitive with the  $\Lambda$  cold dark matter ( $\Lambda$ CDM) model [25] to explain observational properties of DM at cosmological and astrophysical

levels. In particular, a scalar boson field with an extremely small mass of about  $10^{-22}$  eV can explain the cosmological evolution of the universe [11–24], the rotation curves of galaxies [26], the central-density profile of low-surface-brightness galaxies [27], the size of galactic halos [28], and the amount of substructures in the universe [29]. In Ref. [30] it was shown that a complex and self-interacting scalar boson field with a more realistic mass of about 1 eV in a BEC is also a viable DM candidate. Moreover, in this model no fine tuning of the scalar-field energy density at early times is required and the condensate formation is due to self-interactions.

Indeed, BECs are of interest in the context of quantum gravity. In Refs. [42,43] some of us have shown that Planck-scale deformations of the energy–momentum relation that naturally emerges in many quantum-gravity theories (for an excellent nontechnical overview see Ref. [44]) may affect the properties of low-temperature BECs. In particular it was shown that a Planck-scale induced deformation of the Minkowski energy–momentum dispersion relation  $E \simeq \sqrt{m^2c^4 + p^2c^2} + \xi mcp/2M_p$ , where  $m$  is the mass of the bosons,  $M_p$  the Planck mass and  $\xi$  a dimensionless parameter, produces a shift in the condensation temperature  $T_c$  of about  $\Delta T_c/T_c^0 \simeq 10^{-6}\xi_1$  in typical BECs such as  $^{87}\text{Rb}$  [31],  $^7\text{Li}$  [32],  $^{23}\text{Na}$  [33],  $^1\text{H}$  [34],  $^{85}\text{Rb}$  [35],  $^4\text{He}$  [36],  $^{41}\text{K}$  [37],  $^{133}\text{Cs}$  [38], and  $^{52}\text{Cr}$  [39]. The quantum gravity induced shift in  $T_c$  makes possible to upper-bound the deformation parameter as  $|\xi| \lesssim 10^4$  with recent ultra-precise measurements of  $T_c$  as, e.g., in  $^{39}\text{K}$  [40]. In Ref. [42,43] it is also discussed how to enlarge  $\Delta T_c/T_c^0$  thus improving the bound on  $\xi$  and hence realize an *ad hoc* experiment accomplish this. Finally, the Planck-scale induced shift in  $T_c$  is compared with similar effects due to interboson interactions and finite-size effects. These results open a new possibility

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for a quantum gravity phenomenology based on low-temperature condensates, so that BECs truly appear to be a frontier interdisciplinary research field open to many applications.

We also stress how the effect of interactions as well as of finite-size effects might have observable effects on laboratory BE condensates. For example, in Ref. [40] the effect of interactions has been observed in  $^{39}\text{K}$  and a shift in  $T_c$  measured as a function of the interboson  $s$ -wave scattering length  $a$  and data have been fitted with the second-order polynomial  $\Delta T_c/T_c^0 \simeq b_1(a/\lambda_T) + b_2(a/\lambda_T)^2$  with  $b_1 = -3.5 \pm 0.3$  and  $b_2 = 46 \pm 5$ , the second term being due to beyond-mean-field effects. However, in what follows we do not consider interaction nor finite-size effects as we focus on the ideal Bose gas. Such effects are being investigated.

Here we study the metastability of a BEC that does not contain antibosons. A motivation is given in Section 2. In Ref. [41] the properties a RIBG in terms of the Helmholtz free energy with antibosons included was discussed and shown to be a state with a lower Helmholtz free energy than that without antibosons. In Section 3 we generalize this result by comparing two different BECs, with and without antibosons, but with the same total number of particles and at the same finite temperature. Such states are related by a thermodynamic transformation ensuring that they are meaningfully comparable. In particular, we rely on the law of nondecreasing entropy for isolated systems. In Section 4 we conclude that the state with antibosons has greater entropy and lower Helmholtz free energy and which is therefore the stable state, while the state without antibosons is metastable. In Section 5 we derive the expression of the pressure of the BEC in equilibrium with a thermalized gas of bosons and show that the state with antibosons has higher pressure. As an overall check we calculate the Gibbs potential in both cases and show that it is the same, as expected. Lastly, in Section 6 we generalize results for arbitrary  $d > 0$  dimensions, integer or not. We conclude in Section 7.

## 2. Motivation

To study the relative stability of the two states with and without antibosons one should compare their entropies. For a meaningful comparison the two states must be at the same temperature, volume and number density. This is guaranteed in what follows. Consider a system composed of two heat reservoirs  $R_1$  and  $R_2$  at temperatures  $T_1$  and  $T_2$ , respectively, with  $T_1 \ll T_2$ , and a gas of  $N$  bosons  $B$  of mass rest mass  $m$  contained in a volume  $V$  with a number density  $n \equiv N/V$ . The reservoirs are much larger in volume than the boson volume  $V$  so they can be placed in thermal contact with the boson gas without appreciably changing its temperature, or they can be isolated from the boson gas. Assume that  $T_1 < T_c^B < T_c^{B\bar{B}}$ , where  $T_c^B$  and  $T_c^{B\bar{B}}$  are the boson gas BEC critical temperatures without and with antibosons  $\bar{B}$ , respectively, so that at  $T_1$  the boson gas itself is a BEC. Assume also that  $k_B T_2 \gg mc^2$ , with  $c$  the velocity of light, and that  $k_B T_1 \ll mc^2$ . Initially, the gas contains only bosons and is in thermal equilibrium with the first reservoir  $R_1$  at temperature  $T_1$ . Since  $k_B T_1 \ll mc^2$  at this temperature any antibosons present are negligible. The boson gas is then isolated from the reservoir  $R_1$  and placed in thermal contact with the reservoir  $R_2$ . After awhile the boson gas reaches thermal equilibrium at temperature  $T_2$ . Since  $k_B T_2 \gg mc^2$  antibosons are created substantially by pair production so that the equilibrium state now also contains antibosons  $\bar{B}$ . Finally, the boson gas is isolated from the reservoir  $R_2$  and placed in thermal equilibrium with the reservoir  $R_1$  so that the final temperature of the boson gas is  $T_1$ . The question concerning the metastability of the state without antibosons can be formulated in the following way: at the end of the process just described does the boson gas contain antibosons or does it go back to the initial state without antibosons? To answer this one must calculate the entropy variation  $\Delta S_{tot}^I$  of the whole

system (boson gas and reservoirs) a final state of the boson gas with antibosons, and compare it with the entropy  $\Delta S_{tot}^{II}$  of a final state without antibosons. This question is addressed and resolved in Section 4 where we show that  $\Delta S_{tot}^I > \Delta S_{tot}^{II}$  so that the state without antibosons is metastable.

We first calculate the main thermodynamic functions in both cases, with and without antibosons.

## 3. Energy density and Helmholtz free energy below BEC $T_c$

We consider two gas systems, one with only bosons  $B$  and a second one containing also antibosons  $\bar{B}$ . They are both at the same temperature  $T < T_c^B < T_c^{B\bar{B}}$ , where  $T_c^B$  and  $T_c^{B\bar{B}}$  are the condensation temperatures of these two systems without and with antibosons, respectively. We first write down explicit expressions for internal energies and number densities and then proceed to calculate their Helmholtz free energy.

Since  $T < T_c^B < T_c^{B\bar{B}}$  the condensate forms in both a system containing only bosons  $B$  as well as in a system containing also antibosons  $\bar{B}$ . At such temperatures  $T$  the chemical potential  $\mu \simeq mc^2$  in a RIBG whose energy  $E(p)$ –momentum  $p$  dispersion is  $E(p) \equiv \sqrt{p^2 c^2 + m^2 c^4}$ . For a gas containing only bosons the number density is

$$n = n_0 + (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \frac{1}{\exp[\beta(E(p) - mc^2)] - 1} \quad (1)$$

where  $\beta \equiv 1/k_B T$ . The net internal energy per unit volume  $V$  is

$$\frac{U^B(n, T, V)}{V} = mc^2 n_0 + (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \frac{E(p)}{\exp[\beta(E(p) - mc^2)] - 1}. \quad (2)$$

Here

$$n_0 \equiv \frac{1}{V} \frac{1}{\exp[\beta(mc^2 - \mu)] - 1}. \quad (3)$$

Combining these equations leaves

$$\frac{U^B(n, T, V)}{V} = mc^2 n + (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \frac{E(p) - mc^2}{\exp[\beta(E(p) - mc^2)] - 1}. \quad (4)$$

When antibosons are included the number density  $n$  is

$$n = n_0 + (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \left[ \frac{1}{\exp[\beta(E(p) - mc^2)] - 1} - \frac{1}{\exp[\beta(E(p) + mc^2)] - 1} \right] \quad (5)$$

so that

$$\begin{aligned} \frac{U^{B\bar{B}}(n, T, V)}{V} &= mc^2 n_0 + (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp E(p) \\ &\times \left[ \frac{1}{\exp[\beta(E(p) - mc^2)] - 1} \right. \\ &\left. + \frac{1}{\exp[\beta(E(p) + mc^2)] - 1} \right]. \end{aligned} \quad (6)$$

Combining these two equations gives

$$\frac{U^{B\bar{B}}(n, T, V)}{V} = mc^2 n + (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \times \left[ \frac{E(p) - mc^2}{\exp[\beta(E(p) - mc^2)] - 1} + \frac{E(p) + mc^2}{\exp[\beta(E(p) + mc^2)] - 1} \right]. \quad (7)$$

The Helmholtz free energy per unit volume without antibosons is then

$$F^B(T, V, n)/V = mc^2 n + k_B T (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \times \ln[1 - \exp(\beta[mc^2 - E(p)])]. \quad (8)$$

In the case with antibosons one has

$$F^{B\bar{B}}(T, V, n)/V = mc^2 n + k_B T (\hbar^3 2\pi^2)^{-1} \times \int_{0^+}^{\infty} p^2 dp \{ \ln[1 - \exp(\beta[mc^2 - E(p)])] + \ln[1 - \exp(-\beta[mc^2 + E(p)])] \}. \quad (9)$$

From (8)–(9) it also follows that

$$F^{B\bar{B}}(T, V, n) - F^B(T, V, n) = V \frac{k_B T}{\hbar^3 2\pi^2} \int_{0^+}^{\infty} p^2 dp \ln[1 - \exp(-\beta[mc^2 + E(p)])] < 0. \quad (10)$$

Therefore the state containing antibosons has a lower Helmholtz free energy. This same result was found in Ref. [41] except that here the Helmholtz free energies are compared at the same temperature  $T$ .

#### 4. Entropy

Here we calculate the entropy of the boson field with and without antibosons. This is then used to determine the entropy variation in the thermodynamic transformation described in Section 2 to conclude that the state containing antibosons is the stable state while the state without antibosons is only metastable. If only  $B$  bosons are considered, the entropy follows from

$$TS^B(T, V, n) = U^B(T, V, n) - F^B(T, V, n) \quad (11)$$

where the internal energy per unit volume is given by (4). Whence

$$S^B(T, V, n)/V = k_B (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \left\{ \frac{\beta(E(p) - mc^2)}{\exp[\beta(E(p) - mc^2)] - 1} - \ln[1 - \exp(\beta[mc^2 - E(p)])] \right\}. \quad (12)$$

If antibosons are included the entropy follows from

$$TS^{B\bar{B}}(T, V, n) = U^{B\bar{B}}(T, V, n) - F^{B\bar{B}}(T, V, n) \quad (13)$$

where the Helmholtz free energy is given by (9). Using the latter and (7) one gets

$$S^{B\bar{B}}(T, V, n)/V = k_B (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \left\{ \frac{\beta(E(p) - mc^2)}{\exp[\beta(E(p) - mc^2)] - 1} + \frac{\beta(E(p) + mc^2)}{\exp[\beta(E(p) + mc^2)] - 1} - \ln[1 - \exp(\beta[mc^2 - E(p)])] - \ln[1 - \exp(-\beta[mc^2 + E(p)])] \right\}. \quad (14)$$

One can now compare the entropies of the two states with and without antibosons. From (12) and (14) one easily finds that

$$S^{B\bar{B}}(T, V, n) - S^B(T, V, n) = k_B V (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \left\{ \frac{\beta(E(p) + mc^2)}{\exp[\beta(E(p) + mc^2)] - 1} - \ln[1 - \exp(-\beta[mc^2 + E(p)])] \right\} > 0 \quad (15)$$

so that the state *without* antibosons being less entropic is thus metastable.

Now consider the thermodynamic transformation described in Section 2 and calculate the total entropy variation of the boson field plus that of the two reservoirs. This enables one to decide if the final state will contain or not antibosons. The final state of the whole system (boson field plus reservoirs) turns out to be more entropic one which in turn implies that the state with antibosons is the stable state while the state without antibosons is only metastable. If the final state of the gas also contains antibosons, the entropy variation of the gas in the thermodynamic transformation described in Section 2 is

$$\Delta S_{gas}^I = S^{B\bar{B}}(T_1, V, n) - S^B(T_1, V, n) > 0 \quad (16)$$

(which is the same as (15)), while the entropy variation of the two reservoirs is

$$\Delta S_1^I = \frac{\Delta Q_1}{T_1} = \frac{U^{B\bar{B}}(T_2, V, n) - U^B(T_1, V, n)}{T_1} > 0 \quad (17)$$

$$\Delta S_2^I = \frac{\Delta Q_2}{T_2} = \frac{U^{B\bar{B}}(T_1, V, n) - U^{B\bar{B}}(T_2, V, n)}{T_2} < 0. \quad (18)$$

Hence, the total entropy variation is

$$\begin{aligned} \Delta S_{tot}^I &= \Delta S_{gas}^I + \Delta S_1^I + \Delta S_2^I \\ &= \Delta S_{gas}^I + U^{B\bar{B}}(T_2, V, n) \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \\ &\quad + \frac{U^{B\bar{B}}(T_1, V, n)}{T_2} - \frac{U^B(T_1, V, n)}{T_1} \\ &> \Delta S_{gas}^I + [U^{B\bar{B}}(T_2, V, n) - U^B(T_1, V, n)] \left( \frac{1}{T_1} - \frac{1}{T_2} \right) \\ &> 0. \end{aligned} \quad (19)$$

The transformation is thus allowed but is irreversible.

The net entropy variation in the thermodynamic transformation of Section 2 when the final state is without antibosons is thus

$$\Delta S_{gas}^{II} = 0 \quad (20)$$

while the entropy variation of the two reservoirs is

$$\Delta S_1^{II} = \frac{U^{B\bar{B}}(T_2, V, n) - U^B(T_1, V, n)}{T_1} > 0 \quad (21)$$

$$\Delta S_2^{\text{II}} = \frac{U^B(T_1, V, n) - U^{B\bar{B}}(T_2, V, n)}{T_2} < 0. \quad (22)$$

Hence, the total entropy variation is

$$\Delta S_{\text{tot}}^{\text{II}} = [U^{B\bar{B}}(T_2, V, n) - U^B(T_1, V, n)] \left( \frac{1}{T_1} - \frac{1}{T_2} \right) > 0. \quad (23)$$

Again, the transformation is allowed but is irreversible and its only effect is a heat transfer between the two reservoirs  $R_1$  and  $R_2$ .

We can now compare the two entropy variations  $\Delta S_{\text{tot}}^{\text{I}}$  and  $\Delta S_{\text{tot}}^{\text{II}}$ . One has

$$\begin{aligned} \Delta S_{\text{tot}}^{\text{I}} - \Delta S_{\text{tot}}^{\text{II}} \\ = \Delta S_{\text{gas}} + \frac{U^{B\bar{B}}(T_1, V, n) - U^B(T_1, V, n)}{T_2} > 0 \end{aligned} \quad (24)$$

so that  $\Delta S_{\text{tot}}^{\text{I}} > \Delta S_{\text{tot}}^{\text{II}}$ , i.e., the entropy variation is greater in the case in which the final state contains antibosons. Therefore, the final state of the whole system of the gas plus the two reservoirs is more entropic when the gas contains antibosons in the final state. Again this means the state without antibosons is metastable and that the final equilibrium state described in Section 2 is the one containing antibosons.

We remark that if the boson field were a *real* scalar field it would not admit antibosons and in this instance the state containing only bosons  $B$  is the only possible one and therefore it is not metastable but rather a stable state. One thus concludes that if antibosons are allowed, namely if the scalar field is *complex* as assumed here, the state with antibosons is allowed and this state will be the stable one while the state without antibosons will only be metastable.

## 5. Pressure

Following the same procedure we introduce the pressure  $P$  as a function of  $n$ ,  $V$  and  $T$ . Specifically, if no antibosons are present

$$\begin{aligned} \beta V P^B = -\ln[1 - \exp[\beta(\mu - mc^2)]] - (\hbar^3 2\pi^2)^{-1} V \int_{0^+}^{\infty} p^2 dp \\ \times \ln[1 - \exp[\beta(\mu - E(p))]]. \end{aligned} \quad (25)$$

We rewrite (1) as

$$n_0 = n - n_+ = \frac{1}{V} \frac{\exp[\beta(\mu - mc^2)]}{1 - \exp[\beta(\mu - mc^2)]} \quad (26)$$

where

$$n_+ \equiv (\hbar^3 2\pi^2)^{-1} \int_{0^+}^{\infty} p^2 dp \frac{1}{\exp[\beta(E(p) - mc^2)] - 1} \quad (27)$$

is the number density of noncondensate (or excited) bosons. Below the condensation temperature  $\mu \simeq mc^2$  so that  $\exp[\beta(\mu - mc^2)] \simeq 1$  apart from small corrections  $O(1/V)$  which vanish in the thermodynamic limit  $V \rightarrow \infty$ . We can write the logarithm in the rhs of (25) as

$$\ln[1 - \exp[\beta(\mu - mc^2)]] \simeq -\ln[V(n - n_+)] \quad (28)$$

whence

$$\beta V P^B = -\ln \left[ V \left( n - \frac{1}{(\hbar^3 2\pi^2)} \int_{0^+}^{\infty} p^2 dp \frac{1}{\exp[\beta(E(p) - mc^2)] - 1} \right) \right]$$

$$- \frac{V}{(\hbar^3 2\pi^2)} \int_{0^+}^{\infty} p^2 dp \ln[1 - \exp[\beta(mc^2 - E(p))]]. \quad (29)$$

Dividing through by  $V$  gives the first term on the rhs proportional to  $V^{-1} \ln V$  which also vanishes in the thermodynamic limit, so one gets

$$P^B = - \frac{k_B T}{(\hbar^3 2\pi^2)} \int_{0^+}^{\infty} p^2 dp \ln[1 - \exp[\beta(mc^2 - E(p))]]. \quad (30)$$

When antibosons are included the pressure is given by the relation

$$\begin{aligned} \beta V P^{B\bar{B}} \\ = -\ln \left[ V n - \frac{1}{\hbar^3 2\pi^2} \int_{0^+}^{\infty} p^2 dp \left( \frac{1}{\exp[\beta(E(p) - mc^2)] - 1} \right. \right. \\ \left. \left. - \frac{1}{\exp[\beta(E(p) + mc^2)] - 1} \right) \right] \\ - \frac{V}{(\hbar^3 2\pi^2)} \int_{0^+}^{\infty} p^2 dp [\ln(1 - \exp[\beta(mc^2 - E(p))]) \\ + \ln(1 - \exp[-\beta(mc^2 + E(p))])]. \end{aligned} \quad (31)$$

Since the first term in (31) is negligible in the thermodynamic limit, the final result is

$$\begin{aligned} P^{B\bar{B}} = - \frac{k_B T}{(\hbar^3 2\pi^2)} \int_{0^+}^{\infty} p^2 dp (\ln[1 - \exp[\beta(mc^2 - E(p))]] \\ + \ln[1 - \exp[-\beta(mc^2 + E(p))]]). \end{aligned} \quad (32)$$

Comparing (30) with (32) it becomes evident that when antibosons are included the pressure is greater than in the case without antibosons. Indeed, one has

$$\begin{aligned} P^{B\bar{B}} - P^B = - \frac{k_B T}{(\hbar^3 2\pi^2)} \int_{0^+}^{\infty} p^2 dp \ln[1 - \exp[-\beta(mc^2 + E(p))]] \\ > 0. \end{aligned} \quad (33)$$

As a final overall check, comparison of Eqs. (10) and (33) shows that the state with and without antibosons have the same Gibbs free energy  $G(P, T) = F + PV = \mu N$ , namely  $G^{B\bar{B}}(P, T) = G^B(P, T)$ , as must be the case since the net number of particles  $N$  and chemical potential  $\mu$  are the same.

## 6. Generalization to $d$ spatial dimensions

Here we generalize the thermodynamic potentials to arbitrary  $d > 0$  spatial dimensions, integer or not. The result is confirmed that the state with only bosons is metastable while the state with both bosons and antibosons is stable. To motivate this section we recall that spaces with dimensionality different from  $d = 3$  are considered in many physical contexts, e.g., in quantum gravity (see Ref. [44] for a review). In other areas, e.g., Mandelbrot (Ref. [45], p. 85) cites an empirical fractal dimension  $d = 1.23$  for the distribution of galaxies in the observable universe.

We first calculate the thermodynamic functions with and without antibosons. Assuming a real nonnegative number  $d$  of spatial dimension, the sum over momentum now becomes

$$\sum_{\mathbf{p} \neq 0} \rightarrow \left(\frac{L}{2\pi\hbar}\right)^d \Omega_d \int_{0^+}^{\infty} p^{d-1} dp \quad (34)$$

where  $\Omega_d$  is the solid angle in  $d$  dimensions and the system volume is  $L^d$ .

If no antibosons are present the number density  $n$  is

$$n = n_0 + \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp [\exp[\beta(E(p) - \mu)] - 1]^{-1} \quad (35)$$

where

$$n_0 \equiv [V (\exp[\beta(mc^2 - \mu)] - 1)]^{-1}. \quad (36)$$

As before,  $n_0 \equiv N_0/L^d$  is the number density of zero-momentum  $\mathbf{p} = 0$  bosons within a  $d$ -dimensional volume  $V \equiv L^d$ . The internal energy per unit volume is

$$U^B(T, V, n)/V = nmc^2 + \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \frac{E(p) - mc^2}{\exp[\beta(E(p) - \mu)] - 1}. \quad (37)$$

The Helmholtz free energy is

$$F^B(T, V, n)/V = mc^2 n + k_B T \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \times \ln[1 - \exp[\beta(\mu - E(p))]] \quad (38)$$

while the entropy per unit volume is now

$$S^B(T, V, n)/V = k_B \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \left\{ \frac{\beta(E(p) - mc^2)}{\exp[\beta(E(p) - \mu)] - 1} - \ln[1 - \exp[\beta(\mu - E(p))]] \right\} \quad (39)$$

with the term  $(k_B/V) \ln[1 - \exp[\beta(\mu - mc^2)]]$  being negligible in the thermodynamic limit.

If antibosons are present the number density is

$$n = n_0 + \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \left[ \frac{1}{\exp[\beta(E(p) - \mu)] - 1} + \frac{1}{\exp[\beta(E(p) + \mu)] - 1} \right] \quad (40)$$

where  $n_0$  is still (36). The internal energy per unit volume is

$$\frac{U^{B\bar{B}}(n, T, V)}{V} = mc^2 n + \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \left[ \frac{E(p) - mc^2}{\exp[\beta(E(p) - mc^2)] - 1} + \frac{E(p) + mc^2}{\exp[\beta(E(p) + mc^2)] - 1} \right]. \quad (41)$$

The Helmholtz free energy per unit volume is

$$F^{B\bar{B}}(T, V, n)/V = mc^2 n + k_B T \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \times \{ \ln[1 - \exp[\beta(mc^2 - E(p))]] - \ln[1 - \exp[-\beta(mc^2 + E(p))]] \}. \quad (42)$$

Finally, the entropy per unit volume becomes

$$S^{B\bar{B}}(T, V, n)/V = k_B \Omega_d (2\pi\hbar)^{-d} \int_{0^+}^{\infty} p^{d-1} dp \left\{ \frac{\beta(E(p) - mc^2)}{\exp[\beta(E(p) - mc^2)] - 1} + \frac{\beta(E(p) + mc^2)}{\exp[\beta(E(p) + mc^2)] - 1} - \ln[1 - \exp[\beta(mc^2 - E(p))]] - \ln[1 - \exp[-\beta(mc^2 + E(p))]] \right\}. \quad (43)$$

At this point it is easy to generalize to arbitrary  $d$  the result that the state with only bosons is stable, just generalizing (24). For example, by use of (37)–(39), (41)–(43) one easily generalizes (10) and obtains

$$F^{B\bar{B}}(T, V, n) - F^B(T, V, n) = \frac{V_d \Omega_d k_B T}{(2\pi\hbar)^d} \int_{0^+}^{\infty} p^{d-1} dp \ln[1 - \exp(-\beta[mc^2 + E(p)])] < 0 \quad (44)$$

for the difference of the Helmholtz potential. Again, for arbitrary  $d$  the state containing antibosons has a lower Helmholtz free energy. Then one can generalize (15) and obtain

$$S^{B\bar{B}}(T, V, n) - S^B(T, V, n) = \frac{V_d \Omega_d k_B}{(2\pi\hbar)^d} \int_{0^+}^{\infty} p^{d-1} dp \left\{ \frac{\beta(E(p) + mc^2)}{\exp[\beta(E(p) + mc^2)] - 1} - \ln[1 - \exp(-\beta[mc^2 + E(p)])] \right\} > 0 \quad (45)$$

so that the state with antibosons is more entropic also for arbitrary  $d$ .

Proceeding in the same way one verifies that the relations resumed in Eqs. (15)–(23) are still valid and therefore Eq. (24) is also valid for arbitrary  $d$ . Therefore one concludes that, also for arbitrary  $d$ , the state without antibosons is metastable.

### 7. Conclusions

The metastability of a Bose–Einstein condensate (BEC) that does not contain antibosons was studied for the relativistic ideal Bose gas (RIBG). In particular, the Helmholtz free energy with both bosons and antibosons was shown to be a state with a lower Helmholtz potential than that without antibosons. This was done with the same number of particles and at the same finite temperature below the BEC critical temperature. Both states were found to be related by a thermodynamic transformation ensuring that they are meaningfully comparable. In addition, relying on the principle of nondecreasing entropy for isolated systems we found that the state with antibosons has greater entropy and is therefore the stable state, while the state without antibosons is metastable. The pressure of both systems was calculated and found to be higher for the state with antibosons than for the state without them. We also confirm that the two states with and without antibosons have

the same Gibbs free energy, as expected. Lastly, results were generalized for arbitrary dimensions  $d > 0$ , integer or not.

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