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STABILITY OF THIN LIQUID FILMS FALLING DOWN ISOTHERMAL AND NONISOTHERMAL WALLS

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This paper reviews important results found in the past years on thin films falling down isothermal and nonisothermal walls. The discussion on isothermal flows is presented as the basis and background for the study of nonisothermal flows. Different model equations are presented and their approximations are discussed. Both linear and nonlinear results are surveyed on uniform and nonuniform heating of the wall. Also a review is given of the effect the curvature of the wall has on flows down vertical cylinders.

KEY WORDS: *thin films down walls, thermocapillarity, thermal Marangoni, model equations, nonuniform heating, flows down cylinders*

1. INTRODUCTION

The behavior of thin liquid films has been investigated for many years. Their research has important consequences in industrial applications under gravity and microgravity situations. Those films under microgravity are important in material crystallization research in space. In earth conditions, the films are subjected to the force of gravity, which may play the double role of stabilizing and destabilizing agent, depending on the dynamic conditions. In the case of thin films flowing down walls, gravity force plays the role of the pressure gradient as the source of motion.

Important applications of liquid films include those of coating of walls, cooling of heated mechanical or electronic systems, and the flow of bubbles in a tube. One of the problems of coating is that the finishing of the surface should be very smooth after the fluid films solidify. This is of interest, for example, in the process of spin coating. When liquid films are used for cooling, the free surface needs to show deformations to increase the effective area for high heat dissipation and, therefore, instability is expected. However, applications extend also to biological systems, for example, to the flow of a nonisothermal liquid layer over the eye and the flow of pulmonary airway Newtonian or viscoelastic lining. Applications are also found in microfluidics and nanofluidics.

These phenomena have been investigated for many years and have been reviewed in the literature. In this introduction, information is given of previous surveys and books which still have impact in modern research. Thus, the goal is to present a wide panorama to understand that this area yesterday was and today is of very intensive research. The review papers are presented first.

Fulford (1964) presents an extensive survey of a variety of applications of thin films in chemical engineering. Levich and Krylov (1969) analyze problems related with surface tension and its changes with different geometries, emphasizing chemical engineering phenomena.

Thin films flowing down isothermal walls have been reviewed by Lin (1983) and by Lin and Wang (1986), where they present some elements of the linear and nonlinear theory. Vorontsov (1993) reexamines different theoretical and experimental results related with transition and turbulence in thin films and proposes probability and statistics as a way to describe this phenomena. Chang (1994) presents the limitations of different model equations, mainly due to the singularities they show and gives a review of results related with the integral boundary layer equations in two

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NOMENCLATURE

Bi	Biot number	T_{ambient}	ambient atmosphere temperature
c	phase velocity	T_w	wall temperature
d	d_w/h_0	T_0	wall lower face temperature
d_w	wall thickness	U	representative velocity
$h(x,y,t)$	film local thickness	u	velocity x component
$H(x,y,t)$	film local perturbation	v	velocity y component
h_0	unperturbed film thickness	w	velocity z component
H_h	heat transfer coefficient	We	Weber number
k_x	wavenumber x component		
k_y	wavenumber y component	**Greek Symbols**	
k_c	critical wavenumber	β	wall inclination angle
k_m	maximum growth wavenumber	Γ	growth rate
k_s	subcritical wavenumber	Γ_k	Kapitza number
k_f	fluid heat conductivity	Δ	means difference
k_w	wall heat conductivity	ε	wave slope smallness parameter
Ma	Marangoni number	ζ	wall deformation
p	pressure	η	similarity variable
P_p	surface external pressure	κ	thermal diffusivity
Pr	Prandtl number	λ	wavelength
q	fluid flux variable	ν	kinematic viscosity
Q_c	wall to fluid wall conductivities ratio	ρ	fluid density
		σ	surface tension
Re	Reynolds number	Σ	surface tension number
S	scaled surface tension number	ω	frequency of oscillation
T	fluid temperature	Ω	$\omega + i\Gamma$

dimensions. Oron et al. (1997) give an extensive revision of the isothermal and nonisothermal flows of thin films, including other effects as van der Waals forces, temperature dependent viscosity, etc. Myers (1998) uses the high surface tension approximation to survey a variety of problems which are described by evolution equations obtained under the lubrication approximation. Analytical solutions for steady problems are also discussed. Kondic (2003) gives a survey of the instability from the point of view of the lubrication approximation and presents a comparison of theory and experiment, with a discussion of numerical methods.

The lubrication approximation in thin films with a free surface is inspired by the Reynolds theory of lubrication by fluid layers between two walls in relative motion (see Oron et al., 1997). In that theory the walls are assumed to have deformations with small slope in the sides in contact with the fluid. The idea is to use this slope as a representative small parameter to reduce the Navier-Stokes equations into approximate equations which have the possibility of an analytical solution. When the model equations of the thin films reviewed in this paper are obtained under the lubrication approximation, it is assumed that the slope of the free surface deformation is small. That slope is used as a small parameter to simplify the Navier-Stokes equations and the corresponding complex nonlinear free surface boundary conditions into a set of equations solvable recursively to the desired order in the small parameter. This approximation can be applied to a one-layer, a two-layer, or a multilayer fluid system.

The problem of coating has been discussed by Ruschak (1985). More recent research is presented by Weinstein and Ruschak (2004). Youn et al. (2006) survey the premetered coating flows which include curtain, slide, and slot coating. They also present models for these flows.

Davis (1987) reviews results of thermocapillary problems with temperature gradient perpendicular to the liquid layer and for horizontal temperature gradient. An extensive survey has also been done by Zeytounian (1998), who includes very thin layers which are only susceptible to Marangoni convection and not very thin layers that are sensitive to both buoyancy and Marangoni convection. He also presents results of layers flowing down walls. Velarde et al. (2001) and Velarde and Vignes-Adler (2002) give a discussion on the effect of surface deformation and natural convection on thermocapillary nonlinear surface waves. Kalliadasis (2007) analyzes the problem of heated thin films, comparing all the models obtained to date in order to improve the approximation and derive equations that agree completely with the linear results of the Orr-Sommerfeld equation. A review of thermocapillary phenomena with important applications in microelectronics is given by Kabov (2010). This paper presents results of a great number of experiments of films heated locally and driven by surface shear (due to gas flow over the free surface). The problems of wetting and rupture of the film are discussed when the film is subjected to different thermal and surface shear conditions.

Wetting is discussed in de Gennes (1985) from a physico-chemical point of view. Davis (2000) gives a review of interface phenomena including thin films. The case of ultrathin films, with the problem of rupture, is included. O'Brien and Schwartz (2002) discuss the basis of the approximations used to describe thin films analytically. Besides, they include a review of wetting and dewetting in very thin films where the disjoining pressure is effective. Thiele (2007) examines critically, from the point of view of structure formation, the problems of dewetting depending on the characteristics of the substrate. He also presents the effects of themocapillarity. Bonn et al. (2009) review the recent results on wetting and spreading of drops, important in the spin-coating process. The issue of micro- and nanofluidics is discussed along with the effects produced by the three-phase contact line. Craster and Matar's (2009) survey includes wetting and dewetting of thin films under the influence of heat and surfactants. The effect of substrate properties, like topography, compliance, and rotation, are also reviewed.

A full review on the flow on fibers has been done by Quéré (1999). He describes phenomena when the radius of the cylinder is small and throttling effects are important on the stability.

A number of books have been published which describe theoretical and experimental results. Hewitt and Hall-Taylor (1970) discuss theoretical and experimental results of two-phase flows which occur inside tubes. Also, they present phenomena related with core-annular flows where one of the phases may also include another phase.

The isothermal flow is reviewed by Alekseenko et al. (1994), who published a monograph on thin films where the integral boundary layer equations are used in the theoretical discussion. The theory is compared with experimental results which are presented throughout the book. Chang and Demekhin (2002) present the linear theory before discussing the nonlinearity of surface phenomena of thin films flowing down walls. The analysis follows with equations which are derived from balance approximations of the linear and nonlinear terms of the Navier-Stokes equations by means of a small parameter. Ajaev (2012) presents a review of models of thin films and problems related with coating flows and films flowing down walls with fingering. He includes the effects of electric forces, surfactants, structured substrates, and phase change.

Some books survey thermocapillary flow. The problem of Marangoni convection is extensively discussed in the book by Colinet et al. (2001). They include linear and nonlinear phenomena in one and two layers with surface deformation. Similar problems are reviewed in Nepomnyashchy et al. (2002) but they also include phenomena of drops on heated substrates or subjected to the effects of a surfactant. Birich et al. (2003) survey, among other usual things, the parametric excitation of Marangoni convection, the suppression of Rayleigh-Taylor instability with high frequency oscillations, and multilayer problems. The book by Kalliadasis et al. (2012) analyzes the flow of films falling down walls under isothermal and nonisothermal conditions. Of particular interest are the discussions on the different approximations already done to model thin films, as are the lubrication approximation and the integral boundary layer approximation equations, their success and problems in describing experimental results for different parameter ranges. They compare the model equations with those obtained by the new weighted residual method put forward by Ruyer-Quil, and Manneville and improved by Scheid, Ruyer-Quil and Manneville.

Blossey (2012) gives a survey of dewetting, which is very important in coating problems. Relevant applications need the use of non-Newtonian fluids. Therefore, this book is devoted to reviewing the theory and experiments on polymeric flows and in particular viscoelastic flows.

This paper is devoted to reviewing the flow of thin films flowing down walls including thermocapillary flows. Here, the literature published in more recent years is surveyed along with important classical literature. The goal is to review the success and limitations of well-known models published years ago in order to explain the reason why recently a number of researchers have been motivated to look for new thin film models with a broader range of validity of the parameters involved. Besides, a full review is given on thin films flowing down deformed walls and we discuss the influence the wall geometry has on the stability. The more recent references on the problem of nonuniform heating, which has an increasing importance in microelectronics, are likewise reviewed. Moreover, a discussion is given on the effects the thickness and thermo-physical properties of the wall have on the stability. In addition, a survey is presented on the influence the wall curvature has on the stability of films flowing down isothermal and nonisothermal cylindrical walls.

Therefore, the paper is organized as follows. In the next section the nondimensional scaled equations of motion are presented along with the boundary conditions. The linear theory is discussed in Sec. 3. Section 4 is devoted to reviewing important model equations and the approximations needed to their derivation. A survey of nonuniform heating is given in Sec. 5. Section 6 presents a review of the effect the thickness and thermal conductivity of the wall has on the stability of the film. The flow down cylinders is discussed in Sec. 7. Finally, the conclusions are presented in Sec. 8.

2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Here, the Navier-Stokes equations are presented in nondimensional form with scaled independent variables. The equations are given for the isothermal and nonisothermal problems. The boundary conditions are more general and assume that the wall has thickness and finite thermal conductivity. A sketch of the system is given in Fig. 1.

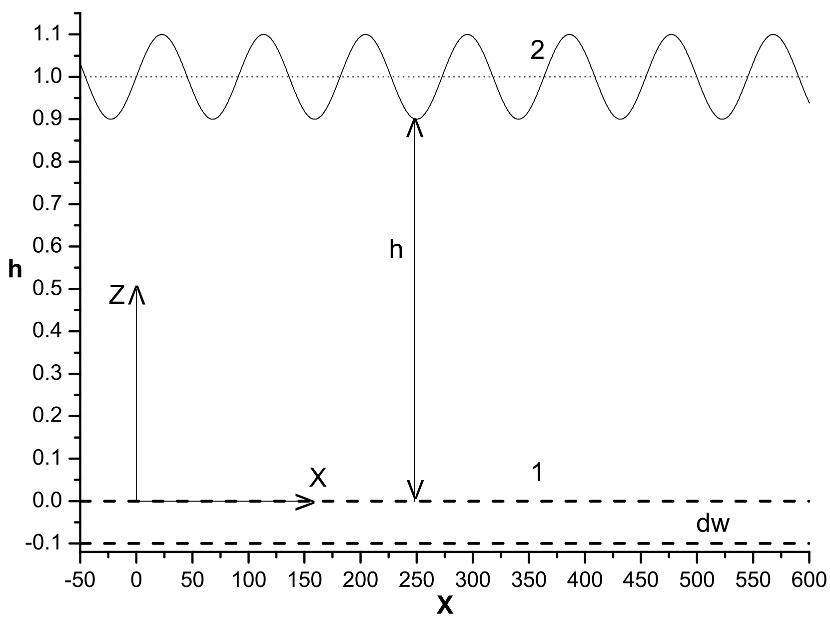

FIG. 1: Sketch of the thin film system. (1) The dashed lines represent the solid wall of nondimensional thickness d_w. The interface between the liquid and the wall is at $z = 0$ and the interface between the wall and the ambient air is at $z = -d_w$. The temperature of the lower side of the wall may be higher or lower than that of ambient atmosphere above the free surface. (2) The solid line is the perturbed free surface. The dotted line at $z = 1$ is the nondimensional height of the unperturbed free surface. h is the local height of the perturbed free surface. A system of reference is plotted with the x and z axes which cross each other at the origin. The x axis is located at the upper side of the wall and the z axis is perpendicular to it. Depending on the angle of inclination of the wall, the figure has to be rotated β degrees.

In the small wavenumber approximation it is assumed that the ratio of the wave amplitude over the representative wavelength is very small. This so-called slope of the wave is represented by the small parameter $\varepsilon = 2\pi h_0/\lambda \ll 1$, where the thickness of the layer is h_0 and λ is the wavelength. Use is made of h_0 to make nondimensional the space variables in the z direction perpendicular to the wall and $\lambda/2\pi$ in the x and y directions, where x is the direction of the main flow down the wall. The nondimensional time is obtained with $h_0\lambda/(2\pi\nu)$. For pressure and velocity, use is made of $\rho\nu^2/h_0^2$ and ν/h_0, respectively. In this way, it is possible to use the parameter ε to scale the equations. Here, ν and ρ are the kinematic viscosity and density of the fluid, respectively. As explained above, the wall has finite thickness and heat conductivity. Thus, the temperature is made adimensional with $\triangle T = (T_0 - T_{\text{ambient}}) > 0$, where T_0 is the temperature at the lower face of the wall and T_{ambient} is the temperature of the ambient atmosphere above the fluid free surface.

In this way, the origin of the system is set at $z = 0$, which is the interface between the wall and the liquid. In nondimensional form, the unperturbed free surface is located at $z = 1$ and the perturbed free surface is at $z = 1 + H(x, y, t) = h(x, y, t)$.

Let us suppose the pressure is p, the velocity components in the (x, y, z) directions are (u, v, w), the temperature is T, and the angle of inclination of the wall is β. The definition of the Reynolds number is $\text{Re} = gh_0^3/\nu^2$, which does not include β. In the nonisothermal case, the Prandtl number is $\text{Pr} = \nu/\kappa$, with κ the heat diffusivity of the liquid. Then, the nondimensional and scaled Navier-Stokes, continuity, and heat diffusion equations are

$$\varepsilon u_t + \varepsilon u u_x + \varepsilon v u_y + w u_z = -\varepsilon p_x + \varepsilon^2 u_{xx} + \varepsilon^2 u_{yy} + u_{zz} + \text{Re}\sin\beta. \tag{1}$$

$$\varepsilon v_t + \varepsilon u v_x + \varepsilon v v_y + w v_z = -\varepsilon p_y + \varepsilon^2 v_{xx} + \varepsilon^2 v_{yy} + v_{zz}. \tag{2}$$

$$\varepsilon w_t + \varepsilon u w_x + \varepsilon v w_y + w w_z = -p_z + \varepsilon^2 w_{xx} + \varepsilon^2 w_{yy} + w_{zz} - \text{Re}\cos\beta. \tag{3}$$

$$w_z = -\varepsilon u_x - \varepsilon v_y. \tag{4}$$

$$\text{Pr}\left(\varepsilon T_t + \varepsilon u T_x + \varepsilon v T_y + w T_z\right) = \varepsilon^2 T_{xx} + \varepsilon^2 T_{yy} + T_{zz}, \tag{5}$$

where the subindexes x, y, z, and t mean partial derivatives. The boundary conditions are evaluated at the lower face of the wall, at the upper face of the wall, and at the free surface. At the upper face of the wall, the non-slip condition is

$$u = v = w = 0, \quad \text{at} \quad z = 0. \tag{6}$$

The normal stress boundary condition is

$$\begin{aligned} -p + \frac{1}{N^2}\left[\varepsilon^3\left(u_x h_x^2 + v_y h_y^2\right) + \varepsilon^3\left(u_y + v_x\right)h_x h_y - \varepsilon\left(v_z + \varepsilon w_y\right)h_y - \varepsilon\left(u_z + \varepsilon w_x\right)h_x + w_z\right] \\ = P_p(x, y, t) - \frac{3}{N^3}S\left[\left(1 + \varepsilon^2 h_y^2\right)h_{xx} + \left(1 + \varepsilon^2 h_x^2\right)h_{yy} - 2\varepsilon^2 h_x h_y h_{xy}\right]. \quad \text{at} \quad z = h(x, y, t), \end{aligned} \tag{7}$$

where $N = \sqrt{1 + \varepsilon^2 h_x^2 + \varepsilon^2 h_y^2}$ and the function $P_p(x, y, t)$ is an external time-dependent pressure applied on the free surface used to control the perturbation frequency. A particular form of $P_p(x, y, t)$ is defined in Eq. (22). The shear stresses are

$$\begin{aligned} \varepsilon(w_z - \varepsilon u_x)h_x - \frac{1}{2}\varepsilon^2(u_y + v_x)h_y + \frac{1}{2}(u_z + \varepsilon w_x)(1 - \varepsilon^2 h_x^2) \\ - \frac{1}{2}\varepsilon^2(\varepsilon w_y + v_z)h_x h_y = \frac{\text{Ma}}{\text{Pr}}(\varepsilon T_x + \varepsilon h_x T_z) \quad \text{at} \quad z = h(x, y, t) \end{aligned} \tag{8}$$

and

$$\begin{aligned} \varepsilon(w_z - \varepsilon u_y)h_y - \frac{1}{2}\varepsilon^2(u_y + v_x)h_x + \frac{1}{2}(v_z + \varepsilon w_y)(1 - \varepsilon^2 h_y^2) \\ - \frac{1}{2}\varepsilon^2(\varepsilon w_x + u_z)h_x h_y = \frac{\text{Ma}}{\text{Pr}}(\varepsilon T_y + \varepsilon h_y T_z). \quad \text{at} \quad z = h(x, y, t). \end{aligned} \tag{9}$$

The temperature conditions are

$$T_w = 1 \quad \text{at} \quad z = -d$$
$$T_w = T \quad \text{and} \quad Q_c dT_w/dz = dT/dz \quad \text{at} \quad z = 0 \tag{10}$$

and

$$T_z + \text{Bi}T = 0 \quad \text{at} \quad z = h(x, y, t), \tag{11}$$

where T_w and T are the temperature distributions in the wall and the fluid, respectively, and Bi = $H_h h_0/k_f$ is the Biot number, which represents the rate of heat flux across the free surface, where H_h is the coefficient of heat transfer and k_f is the fluid heat conductivity. The ratio of the wall and fluid heat conductivities is represented by $Q_c = k_w/k_f$ and the wall and fluid thicknesses ratio is $d = d_w/h_0$.

The kinematic boundary condition is

$$w = \varepsilon h_t + \varepsilon u h_x + \varepsilon v h_y \quad \text{at} \quad z = h(x, y, t). \tag{12}$$

The surface tension changes with temperature. That change depends on the temperature gradient across the film and it is represented by the Marangoni number Ma = $(-d\sigma/dT)\triangle T h_0/(\rho\nu\kappa)$. Here, it is supposed that the liquid has a strong surface tension σ and $\Sigma = \sigma h_0/(3\rho\nu^2)$, the surface tension number, is changed into $S = \varepsilon^2\Sigma$, which is of order one. Note that Σ = We $\text{Re}^2/3$, where We = $\sigma\nu^2/gh_0^5\rho$ is the Weber number.

For the small wavenumber approximation it is necessary to expand the variables in terms of the small parameter ε. To this goal, use is made of the lubrication approximation in which the z component of the velocity is very slow. This is taken into account in the following expansion:

$$u = u_0 + \varepsilon u_1 + \cdots, \quad v = v_0 + \varepsilon v_1 + \cdots, \quad w = \varepsilon(w_1 + \varepsilon w_2 + \cdots),$$
$$p = p_0 + \varepsilon p_1 + \cdots, \quad T = T_0 + \varepsilon T_1 + \cdots, \quad T_w = T_{w0} + \varepsilon T_{w1} + \cdots. \tag{13}$$

Note that it is implicitly supposed that the components of velocity, the pressure, and the temperatures of the fluid and the wall depend on (x, y, z, t), and that only the free surface deformation h depends on (x, y, t).

3. LINEAR THEORY

Here, the linear theory of thin films flowing down inclined walls is presented in the small wavenumber approximation. The equations of motion and boundary conditions are linearized and the variables are separated using normal modes of the form $F(x, y, z, t) = A(z)\exp[i(k_x x + k_y y) - i\Omega t]$. Here, $A(z)$ is the amplitude, k_x and k_y are the x- and y-components of the wavenumber of the perturbation, and $k = \sqrt{k_x^2 + k_y^2}$ is its magnitude. $\Omega = \omega + i\Gamma$, where ω is the frequency of oscillation of the perturbation and Γ is the growth rate. This is used in the expansions Eqs. (13). The free surface has a small perturbation around the location z = 1 of the unperturbed flat surface. It is written in the form $h(x, y, t)$ = 1 + $H(x, y, t)$, where $H(x, y, t)$ = A(1) $\exp[i(k_x x + k_y y) - i\Omega t]$ is the small surface deformation and terms with products of this function or its derivatives are neglected.

3.1 Linear Isothermal Instability

The linear stability in the isothermal case was first investigated by Yih (1963) in the small wavenumber approximation. His result, in our nondimensional form, can be obtained substituting $h(x, y, t)$ = 1 + $A(1)\exp[i(k_x x + k_y y) - i\Omega t]$ in the linearized Benney Equation (21) discussed below and calculated using the lubrication approximation (see Joo et al., 1991a; Dávalos-Orozco et al., 1997). In that equation it is assumed that the fluid surface tension is very large and that $k^2\Sigma$ is of order one, where Σ is the surface tension number defined before.

One result is that the frequency of oscillation is $\omega = k\text{Re}\sin\beta$ (or the phase velocity $c = \omega/k = \text{Re}\sin\beta$ is proportional to Re). Another result is that the growth rate is

$$\Gamma = k^2\left(\frac{2\text{Re}^2\sin^2\beta}{15} - \frac{1}{3}\text{Re}\cos\beta - k^2\Sigma\right). \tag{14}$$

The critical wavenumber is obtained when the growth rate is zero, that is

$$k_c = \sqrt{\frac{1}{\Sigma}\left(\frac{2}{15}\mathrm{Re}^2 \sin^2 \beta - \frac{1}{3}\mathrm{Re} \cos \beta\right)}. \quad (15)$$

Notice that Eq. (14) is calculated linearizing the nonlinear Benney Equation (21), calculated in the small wavenumber approximation. It corresponds to Eq. (37) of Yih (1963), but in a different nondimensional form. He did not present Eq. (15), which gives the critical value at which Γ is zero, that is, k_c is one of the roots of Γ. Observe that, because the frequency is linearly related to the wavenumber by means of $\omega = k\mathrm{Re}\sin\beta$, Γ has a similar relation with ω as with k, but with different coefficients. By taking the derivative of Γ with respect to k equal to zero, it is possible to calculate the wavenumber of maximum growth rate. It can be shown that it satisfies the relation $k_m = k_c/\sqrt{2}$.

The nonlinear Benney equation, which describes the evolution of the free surface deformations of the films and which is given in Eq. (21) of Sec. 4, was used by Gjevic (1970) to calculate a nonlinear subcritical wavenumber k_s below which the solutions of the equation may not saturate. Nonlinear wave saturation means that for some evolution time the wave reaches an amplitude which remains stationary. In this case, the wave ”forgets” about the initial exponential growth in time of the linear theory, which is effective only for a very short time before the nonlinear terms become important.

Gjevic (1970) has shown the nonlinear result $k_s = k_c/2$. The k_s is calculated introducing a normal mode expansion in four terms of the solution $H(x,t) = A_1(t)\exp(ikx) + A_2(t)\exp(2ikx) + cc$ into the Benney equation, where cc means complex conjugate terms. Four coupled first-order ordinary differential equations are obtained for the time-dependent complex amplitudes $A_1(t)$ and $A_2(t)$ and their complex conjugates. Then, they are transformed into $A_1(t) = a_1(t)\exp[i\Theta_1(t)]$ and $A_2(t) = a_2(t)\exp[i\Theta_2(t)]$, where $a_1(t)$, $a_2(t)$, $\Theta_1(t)$, and $\Theta_2(t)$ are real functions. After taking real and imaginary part of the resulting equations a critical condition of the new equations is found for the possibility of a stationary solution, that is, the k_s defined above (see Gjevik, 1970). This condition sets the limitations to the Benney-type equations to represent real physical results, because for perturbations with wavenumber smaller than k_s the numerical solutions may not saturate.

However, note that it has been shown numerically by Joo et al. (1991a) and Davalos-Orozco (1997) that saturation can be found below but near the curve of subcriticality. Despite this result, the curve of subcriticality is a very important reference for the solutions of the Benney-type equations. All these results can be written as

$$k_c = \sqrt{2}k_m = 2k_s. \quad (16)$$

The relation among wavenumbers shown in Eq. (16) is the same for other Benney-type models, as explained below. This might be due to the order taken in the normal mode approximation for the calculation of k_s. However, it is not possible to say that the result is valid for all Benney-type equations. The three curves of k_c, k_m, and k_s were calculated by the author and are plotted in Fig. 2 for different angles of inclination. Notice that the magnitudes of k_m and k_s may be obtained using Eq. (16).

In Fig. 2, above the thin continuous lines of k_c the flow is stable. Between the thin continuous line and the dashed line (for k_c and k_s, respectively) the solutions of the nonlinear Benney equation grow but saturate. If a controlled time-dependent perturbation of frequency ω is applied to the falling film of a given Reynolds number Re, the free surface perturbations will respond with a wavenumber corresponding to that of the crossing point with the thick continuous lines. The Reynolds number where the curves k_c touch the horizontal axis is the minimum critical Reynolds number R_c above which the flow is unstable.

de Bruin (1974) investigated the flow using the numerical solution of the Orr-Sommerfeld equation, where it is assumed that the inclination angles of the wall are very small and that the surface tension σ is zero. He uses a larger range of wavenumbers and defines the soft modes as the longwave (small wavenumber) surface modes and the hard modes as due to shear waves (but here modified by free surface waves) which appear only at small angles of inclination of the wall. The numerical results show that for angles $\beta < 1'$ (less than 1 min.) the stability is governed by the hard mode and large magnitudes of Re $\sin\beta$. In $1' < \beta < 1^\circ$ the soft mode is unstable for smaller Re $\sin\beta$ but the growth rate of the hard mode is an order of magnitude larger in that range. For $\beta > 1^\circ$ the soft mode is important. Moreover, between the angles $3' < \beta < 1^\circ$ an increase in beta decreases the critical Reynolds number, that is, destabilizes.

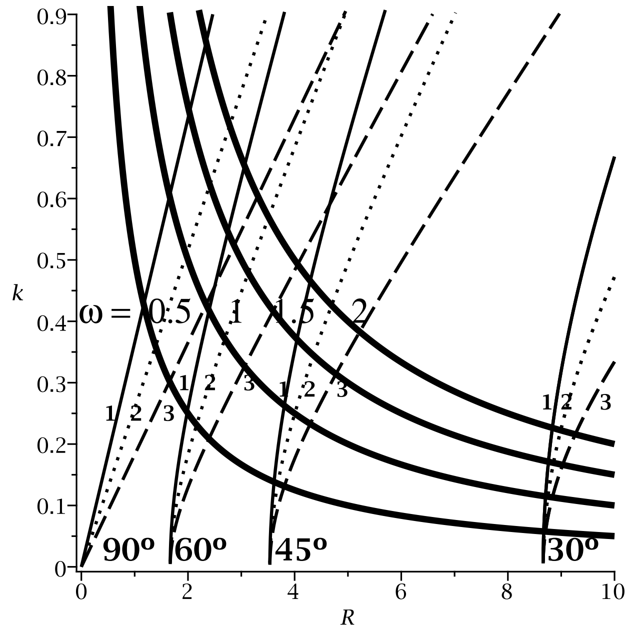

FIG. 2: Isothermal case. Plots of k vs Re for different angles of inclination of the wall 90°, 60°, 45° and 30°. For each angle the numbers are for the curves (1) k_c continuous line [from Eq. (15)], (2) k_m doted line [from Eqs. (15) and (16)], (3) k_s dashed line [from Eqs. (15) and (16)]. The thick continuous lines correspond to the hyperbolas $k = \omega/\text{Re}$ for different frequencies ω = 0.5, 1, 1.5, 2.

The surface tension is very important in thin films and its effect on the above mentioned modes of instability were investigated by Floryan et al. (1987). They found that in the limit of inviscid flow, that is when Re $\sin\beta$ is very large, the hard shear mode is stable for any surface tension number Σ and that the soft surface mode is unstable. They show that the hard mode has a minimum in the curve of the critical $\text{Re}_c \sin\beta$ with respect β and Σ. The growth rates decrease increasing Σ and decreasing β.

Main flow energy as a source of instability was investigated by Smith (1990a). He found two mechanisms of instability. They are explained first by imposing a shear stress on the free deformable surface and second by imposing a velocity at the surface. One conclusion is that when the free surface deforms a perturbation shear stress is produced to restore the original one, and this stress is the source of instability. Another conclusion is that when the interface is deformed a tangential velocity perturbation is produced to recover the original one, and this is the main source of the instability.

Brevdo et al. (1999) investigated the convective and absolute instabilities of a thin film flowing down a wall. They use the full linear Navier-Stokes equations. They found that the instability is convective, in agreement with experiments and the previous results by Joo et al. (1992) for the longwave theory.

The linear stability of thin films flowing down a rotating inclined wall was investigated by Dávalos-Orozco and Ruiz-Chavarría (1992). They found that for the small wavenumber approximation the Coriolis force may stabilize the flow for some angles of propagation of the perturbation and that it can also decrease the growth rate of the perturbation. They made another approximation for small Re, small Taylor numbers (representative of the Coriolis force), and small angles of inclination. In this case, it is shown that an increase of the Taylor number may be destabilizing. The effect of the centrifugal force was included in the paper by Dávalos-Orozco and Busse (2002) in the linear and nonlinear calculations. There, the linear problem was investigated for a large range of rotation rates and wavenumbers. It is shown that an increase of rotation rate changes the small wavenumber as the first unstable mode into one of finite magnitude, phenomena very similar to that of rotating natural convection under fixed heat flux boundary conditions (see Dávalos, 1984). A jump in the stability curves of Re_c and angle of propagation of the perturbation against the

Taylor number appears when this change occurs. The small wavenumber approximation agrees very well with the numerical analysis and generalizes the result of k_c given in Eq. (15). It is also shown that Eq. (16) is still valid. This problem can be extended to flow going upward, as done by Ungarish and Sherwood (2012) for flow in the inside of a rotating cone.

The Benney-type equations are valid for Re $\sim O(1)$. However, the other nonlinear models presented in Sec. 4 are valid for intermediate magnitudes of Re. Those models also have more complex linear equations, as can be seen in Alekseenko et al. (1994) for the integral boundary layer equations. Depending on the magnitudes of We and Re, different approximations of those equations can be done which, by linearization, have different linear equations. For example, in some approximation a two-wave equation is found with linear waves of two different phase velocities. Another approximation, but not the last one, leads to two linear Schrödinger equations. The solutions of those equations have very rich free surface wave phenomena.

3.2 Linear Thermocapillary Instability

The influence of evaporation was taken into account by Bankoff (1971), who found that surface evaporation is destabilizing but condensation is stabilizing. This effect is also taken into account by Joo et al. (1991a). The influence of thermal Marangoni convection in a heated inclined plane was investigated by Lin (1975a), who calculated the linear critical Re_c for instability in the small wavenumber approximation. This critical number depends on the Marangoni number Ma and the Prandtl number Pr. That was later extended to the case of very small angles of inclination of the wall by Sreenivasan and Lin (1978) in order to compare with the horizontal thermocapillary problem where the critical wavenumber is not necessarily small. Their interest was to find out if the flow is stationary or oscillatory. At small Pr, they found the possibility of the oscillatory mode to be the most important and not the stationary one of the usual Marangoni convection. They suggest that the flow structure will be that of rolls in the streamwise (longitudinal) direction.

Kelly et al. (1986) investigated the Marangoni effect in the small wavenumber approximation. They calculate the two roots of Re_c considering Ma but neglecting Σ. They find that one root tends to the isothermal one when Ma $\to 0$ but the other one tends to zero. The first one is called hydrodynamic modified by thermocapillarity and the second one is called thermocapillary modified by shear. When Ma increases, the roots tend to the same magnitude. Then, it is shown that a stability window appears between the first and second roots. Above the magnitude when both Re_c are equal, the flow is always unstable.

Smith (1990b) also made efforts to explain the mechanisms of the thermocapillary instability. He shows that heating or cooling does not change the interfacial instability; it is important when the Prandtl number is large. In this case, buoyancy effects are important in the instability. It is also found that when the free surface is insulating, cooling from below might be destabilizing and heating from below stabilizing.

Goussis and Kelly (1991) found that three mechanisms are important on the stability of a layer flowing down a heated wall. Two are thermocapillary and one is due to shear stress at the deformed free surface. The shear stress mode appears as a transverse wave, while one of the thermocapillary modes prefers longitudinal rolls (when the layer is thick and the temperature gradients are large).

In the present notation and in the small wavenumber approximation, the growth rate and critical wavenumber for the Marangoni effect in a thin film falling down a wall are

$$\Gamma = k^2\left[\frac{2}{15}\mathrm{Re}^2\sin^2\beta - \frac{1}{3}\mathrm{Re}\cos\beta - k^2\Sigma + \frac{1}{6}\frac{\mathrm{Ma}}{\mathrm{Pr}}\frac{\mathrm{Bi}}{(1+\mathrm{Bi})^2}\right]. \tag{17}$$

$$k_c = \sqrt{\frac{1}{\Sigma}\left[\frac{2}{15}\mathrm{Re}^2\sin^2\beta - \frac{1}{3}\mathrm{Re}\cos\beta + \frac{1}{6}\frac{\mathrm{Ma}}{\mathrm{Pr}}\frac{\mathrm{Bi}}{(1+\mathrm{Bi})^2}\right]}. \tag{18}$$

Equation (17) can be obtained linearizing the Benney-type Eq. (23) calculated under the lubrication approximation and using $h(x,y,t) = 1 + A(1)\exp\left[i(k_x x + k_y y) - i\Omega t\right]$ (see Joo et al., 1991a; Dávalos-Orozco, 2012). Equation (18) is the critical condition for a zero growth rate, that is, it is the root of Eq. (17). By taking the derivative of Γ with respect to k equal to zero, it is also possible to calculate k_m for the maximum growth rate in this case. The k_s is calculated in the same way as in the isothermal case, using a normal mode expansion approximation of the Benney-type Eq. (23). It can be shown that also in the thermocapillarity case the curves of criticality, maximum growth rate, and subcriticality satisfy Eq. (16).

Figures 3 and 4 were calculated by the author. The curves presented in Fig. 3 are for the case of a vertical wall, but for different Marangoni numbers and Biot numbers. The important behavior of the coefficient of the thermocapillary term in Eqs. (17) and (18) is shown in Fig. 4 in order to understand the results of Fig. 3. The growth of that coefficient is nonmonotonic and therefore, in the small wavenumber approximation, an increase of Bi not necessarily has a stabilizing effect.

Scheid et al. (2005a,b) test their new model equations for a heated film comparing the linear stability results with those of the Orr-Sommerfeld equation. They found a very good agreement in a large range of values of the parameters, in contrast with the usual small wavenumber approximation, which has some limitations. Comparison is also made with the linear version of other nonlinear models and shows the range of the parameters where they do not agree well with the Orr-Sommerfeld equation.

A binary liquid mixture is a system of two component substances. The Soret effect is the diffusion of a substance promoted by temperature gradients. Therefore, in the case of thin films, the nondimensional Soret number is the ratio of mass diffusivity due to temperature over mass diffusivity divided by the ratio of surface tension variation due to temperature gradients over that due to concentration gradients. The Soret number can be positive or negative. Hu et al. (2008) made extensive calculations of the stability of a thin binary liquid film flowing down a heated wall. It is found that the more important modes are stationary for positive Soret numbers and oscillatory for negative ones. There may be short and long wave instabilities. Hu et al. distinguish among three modes as done by Goussis and Kelly (1991), but with a great variety of phenomena due to the Soret effect.

Samanta (2008) made a small Reynolds number approximation for thermocapillary phenomena in order to generalize the calculations of the isothermal problem. His results agree very well with those of Yih (1963). It is interesting to note the more complex dependence the growth rate has with respect to Bi in comparison with the small wavenumber approximation.

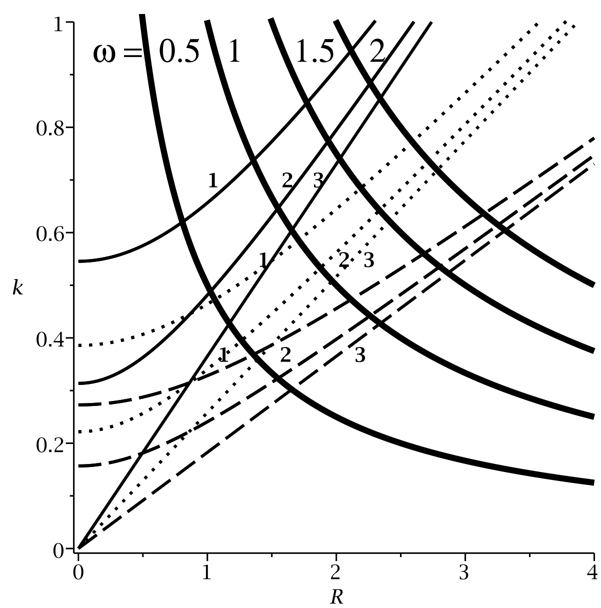

FIG. 3: Nonisothermal case. Vertical wall. Plots of k vs Re for different Ma and different Bi. The thin solid curves are for k_c, the dotted curves are for k_m, and the dashed curves are for k_s. (1) Ma = 10 and Bi = 1, (2) Ma = 10 and Bi = 10, and (3) is the isothermal case. Note that Bi = 10 stabilizes with respect Bi = 1 (see Fig. 4). The thick continuous lines correspond to the hyperbolas $k = \omega/\text{Re}$ for different frequencies ω = 0.5, 1, 1.5, 2.

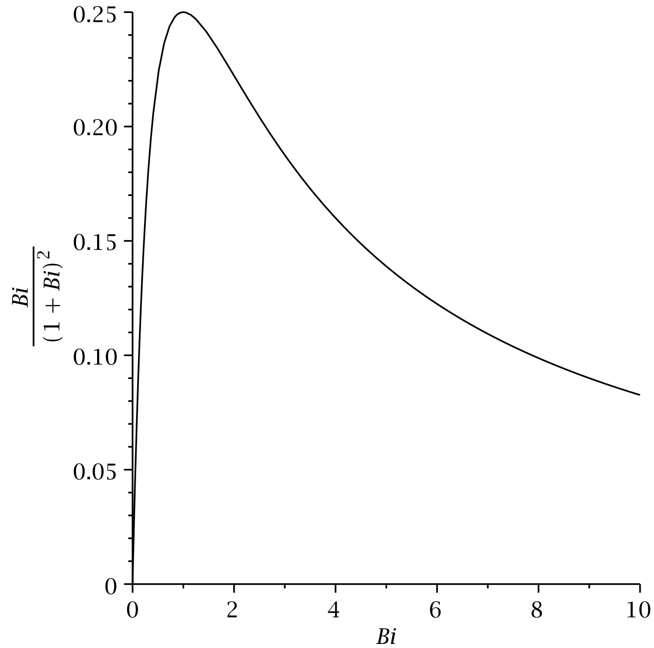

FIG. 4: The coefficient of the thermocapillary term of Eqs. (17) and (18). Its growth is nonmonotonic and has a maximum at Bi = 1. Notice that its effect is the same for Bi = 0.1 as for 10.

In some papers the thickness and thermal conductivity of the wall have been taken into account only for analytical reasons. For example, in order to eliminate the singularities at the rupture point in their problem of thermal and evaporative instabilities, Oron et al. (1996) used the effect of thickness of the wall. Moreover, in order to allow for wall deformations in the side in contact with the liquid film, Kabova et al. (2006) introduced the thickness of the wall. In contrast, Gambaryan-Roisman (2010) used the thermal conductivity and thickness of the wall but her main concern was the effect of the nonuniformity of the thermal conductivity. Besides, Gambaryan-Roisman and Stephan (2009) used the thickness of the wall to allow for wall deformation at the interface between the wall and a rivulet. These papers do not make a systematic research taking into account the effect of the conductivity and thickness of the wall. Dávalos-Orozco (2012) made linear and nonlinear calculations of this problem. He found that the growth rate and the critical wavenumber satisfy the following equations:

$$\Gamma = k^2 \left\{ \frac{2\mathrm{Re}^2 \sin^2 \beta}{15} - \frac{1}{3}\mathrm{Re} \cos \beta - k^2 \Sigma + \frac{1}{6} \frac{\mathrm{Ma}}{\mathrm{Pr}} \frac{\mathrm{Bi}}{\left[1 + \mathrm{Bi} + \mathrm{Bi}\left(\dfrac{d}{Q_c}\right)\right]^2} \right\}. \tag{19}$$

$$k_c = \sqrt{\frac{1}{\Sigma} \left\{ \frac{2}{15}\mathrm{Re}^2 \sin^2 \beta - \frac{1}{3}\mathrm{Re} \cos \beta + \frac{1}{6} \frac{\mathrm{Ma}}{\mathrm{Pr}} \frac{\mathrm{Bi}}{\left[1 + \mathrm{Bi} + \mathrm{Bi}\left(\dfrac{d}{Q_c}\right)\right]^2} \right\}}. \tag{20}$$

Note that the effect of the thickness ratio and conductivities ratio appear in only one parameter d/Q_c. It is also possible to calculate the curve of subcriticality and it is shown that here again k_c, k_m and k_s satisfy Eq. (16). Note that the coefficient of the thermocapillary term is different, as can be seen in Fig. 5. The absolute maximum $1/4$ of

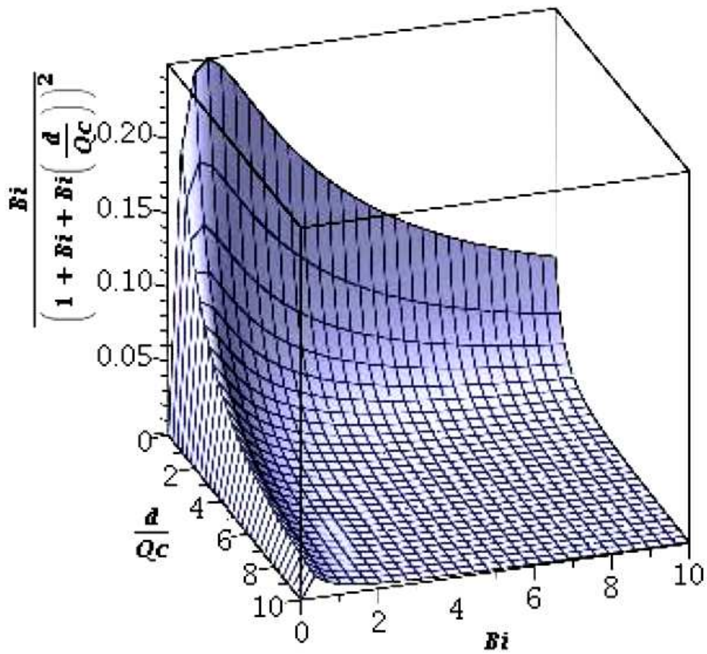

FIG. 5: Three dimensional plot of the coefficient $\mathrm{Bi}/[1 + \mathrm{Bi} + \mathrm{Bi}(d/Q_c)]^2$ of the thermocapillary term when the ratios of thicknesses d and thermal conductivities Q_c are taken into account.

$\mathrm{Bi}/[1 + \mathrm{Bi} + \mathrm{Bi}(d/Q_c)]^2$ is located at Bi = 1 when d/Q_c = 0 but the relative maxima $1/4[1 + (d/Q_c)]$ are at $\mathrm{Bi}_{\max}$ = $1/[1 + (d/Q_c)]$. It is clear that an increase of d/Q_c reduces the magnitude of the coefficient and therefore stabilizes the flow. A sample of curves k vs Re for this problem using Eq. (16) are shown in Fig. 6 for Ma = 50.

In the next section, important model equations will be presented to understand their properties and defects in approximating the real flows.

4. NONLINEAR MODEL EQUATIONS

This section presents a review of different model equations that have important influence in research on films falling down walls. First, the Benney equation is discussed, and next the equation derived by Ooshida. Then, follows the set of equations derived from the integral boundary layer model. Finally, a survey is given of the set of equations obtained by means of the weighted residual boundary layer approximation model and the improvements made in recent years.

4.1 The Benney Equation

The Benney equation (Benney, 1966) is derived by means of a particular scaling where the distances in the direction of the main flow and parallel to the wall are scaled with the perturbation wavelength and that in the direction perpendicular to the wall is scaled with the thickness of the film. After substitution in the equations of motion, energy and boundary conditions a parameter ε appears which is assumed small and represents the adimensional small wavenumber of the perturbation. This scaling was already presented in Sec. 2. Use of the expansions Eqs. (13) in the equations and boundary conditions leads, at first order in the approximation, to the Benney equation. The approximation also assumes that the main velocity and temperature fields are evaluated at the deformed free surface h instead of h_0 (the thickness of the unperturbed layer). The Benney equation is

$$h_t + \mathrm{Re}\sin\beta h^2 h_x + \varepsilon\left\{(\mathrm{Re}\sin\beta)^2\left(\frac{2}{15}h^6 h_x\right)_x + \frac{1}{3}\nabla\cdot\left[h^3\big(-\mathrm{Re}\cos\beta\nabla h + 3S\nabla^2\nabla h - \nabla P_p\big)\right]\right\} = 0. \quad (21)$$

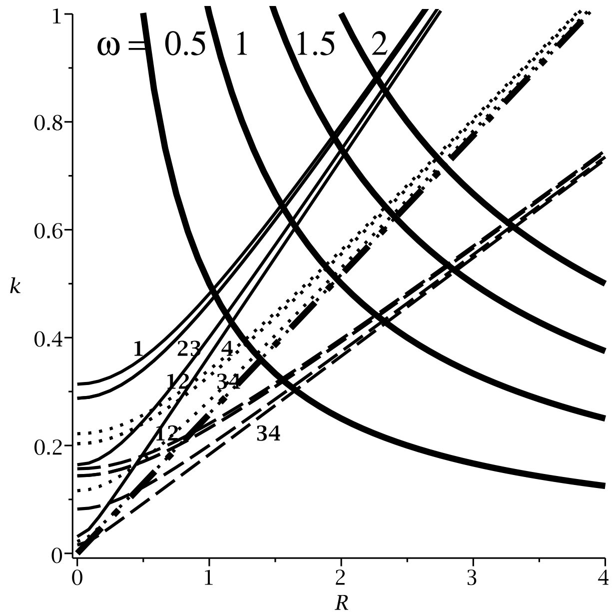

FIG. 6: β = 90°, Pr = 7, Σ = 1, Bi = 0.1, Ma = 50. The solid, dotted, and dashed curves correspond to the curves of criticality, maximum growth rate, and subcriticality, respectively. The straight dot-dashed one is the maximum growth rate of the isothermal problem used as reference. The numbers in the curves indicate (1) d/Q_c = 0.0, (2) d/Q_c = 1, (3) d/Q_c = 10, (4) d/Q_c = 100. The curves of d/Q_c = 0.01 and 0.1 are almost superposed to that of d/Q_c = 0. The thick continuous lines correspond to the hyperbolas $k = \omega/\mathrm{Re}$ for different frequencies ω = 0.5, 1, 1.5, 2.

Benney (1966) obtained an equation which includes terms of higher order in ε. However, he assumes an order one surface tension number (not the scaled one used here) and it appears at the next order, not as in Eq. (21). Even in that case, he is able to show that from different scaling of the parameters it is possible to derive well-known nonlinear equations. For example, the linearized Eq. (21) agrees with the linear equation whose normal modes expression has a growth rate and critical wavenumber Eqs. (14) and (15) calculated by Yih (1963). In some nonlinear approximations Benney (1966) reduced Eq. (21) to the Burgers equation. In another approximation, it can be reduced to the Korteweg and De Vries (KdV) equation. The last one brings about the possibility of solitary wave solutions of Eq. (21) for some parameter range. This possibility has been demonstrated theoretically by Pumir et al. (1983) by means of the theory of dynamical systems in a reference frame moving with the phase velocity, where the wave is stationary.

Gjevik (1970) analyzed Eq. (21) with a normal mode expansion and found that in order to have nonlinear stability the magnitude of the perturbation wavenumber has to be larger than that of the subcritical k_s defined in Eq. (16) in relation with Eq. (15) of k_c. This k_s already gives theoretical limitations to the solutions of the Benney equation. Though Joo et al. (1991a) and Dávalos-Orozco et al. (1997) have shown numerically that it is still possible to attain saturation below but near to k_s (see curve 3 in Fig. 7), this equation is restricted to Reynolds numbers of order one and wavenumbers in the range $k_c \leq k \leq k_s$.

In order to present numerical solutions in space and time of Eq. (21), use is made of the definition

$$P_p(x, y, t) = A\left|\sin\frac{\omega}{2}t\right|\exp[-a(x^2 + y^2)], \tag{22}$$

of the function which appears in the normal stress boundary condition and in Eq. (21). It is assumed that it is an external pressure due to a turbulent air jet which strikes periodically at the origin on the free surface. For applications of a nonoscillating turbulent air jet see Lacanette et al. (2006). The constants which appear in Eq. (22) will be taken as A = 0.0001 and a = 0.05. The selection was made due to the sensitivity of the thin film instability to the parameters.

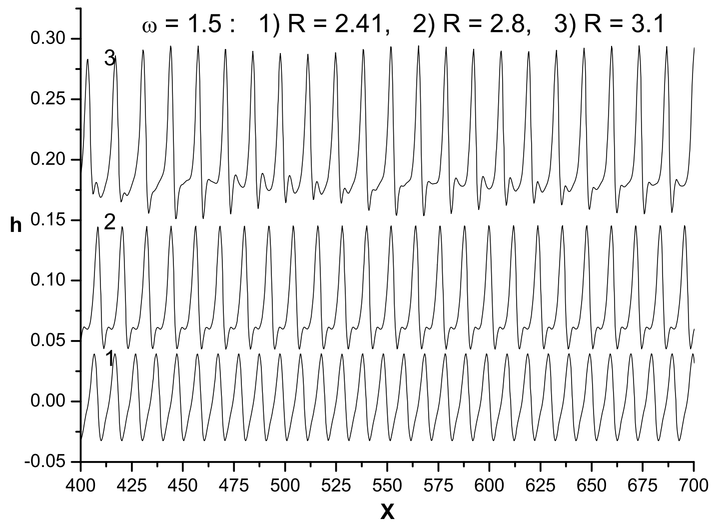

FIG. 7: β = 90°, ω = 1.5, time T = 600. (1) Re = 2.41 (maximum growth rate), (2) Re = 2.8 (above subcriticality), (3) Re = 3.1 (below but near subcriticality). See Fig. 2 as reference. Curves 1 and 2 both show solitary waves but that of curve 2 has space modulation.

The frequency of oscillation ω is divided by two because a jet has no suction and it is effective only when it strikes again on the surface. In this way, numerical solutions in space and time of Eq. (21) are presented in Davalos-Orozco (1997). Some amplified samples in the x interval (400, 700) are given here in Fig. 7 for a film on a vertical wall with a time-dependent perturbation of frequency ω = 1.5. Note that an increase of Re increases the amplitude of the free surface perturbation.

The nonlinear three-dimensional instability of Eq. (21) was investigated by Joo and Davis (1992). They found a secondary instability which leads to three-dimensional waves. Besides, they show that the wavelength depends on the inverse of the square of the layer thickness and that a critical thickness exists below which three-dimensional instability is not possible, for a fixed Re. Lee and Mei (1996) investigated, in the small wavenumber approximation up to second order, the stability of the flow in the framework of dynamical systems. Experimental results obtained by Liu et al. (1993) were compared with the numerical results of Joo et al. (1991a) and Joo and Davis (1992). In another paper Liu and Gollub (1994) found experimentally the appearance of solitary waves and latter Liu et al. (1995) extended the experiments to three-dimensional waves. Joo et al. (1991b) investigated the tendency to wave breaking when the surface tension is not large. A more general equation of the Benney type for viscoelastic and shear thinning fluids has been calculated by Joo (1994). Oron and Gottlieb (2004) investigated the stability of the film using the Benney equation of first and second-order. They found that the Hopf bifurcation of the first order is supercritical for small Re and subcritical for large Re. This is in contrast to the bifurcation of the second-order Benney equation which is supercritical for any Re. They calculate a complex Ginzburg-Landau equation to understand the linear and nonlinear instability. Ali et al. (2005) investigate the rupture of a nanoliquid film. The linear and nonlinear instability is investigated. They find that when the film thickness is thinner than 100 nm, the van der Waals forces are important and that when it is thicker gravity controls the stability.

The nonlinear Benney-type equation with thermocapillary and evaporation effects is investigated numerically by Joo et al. (1991a). In the absence of evaporation, it has the form

$$h_t + \mathrm{Re}\sin\beta h^2 h_x + \varepsilon\Bigg\{(\mathrm{Re}\sin\beta)^2\left(\frac{2}{15}h^6 h_x\right)_x + \frac{1}{3}\nabla\cdot\Bigg[h^3\big(-\mathrm{Re}\cos\beta\nabla h + 3S\nabla^2\nabla h - \nabla P_p\big) + \frac{1}{2}\frac{\mathrm{Ma}}{\mathrm{Pr}}\frac{\mathrm{Bi}h^2}{(1+\mathrm{Bi}h)^2}\nabla h\Bigg]\Bigg\} = 0. \tag{23}$$

Note that only the temperature at the lowest order is needed in this case. It is clear that the linear coefficient of the thermocapillary effect in the equation is the same as that plotted in Fig. 4. Therefore, it is easy to understand why the surface deformation of the film down a vertical wall plotted in Fig. 8 for $\omega = 0.5$ has a larger amplitude for Bi = 1 than for Bi = 10. The maximum of the coefficient is attained at Bi = 1. The value Ma = 0 means isothermal calculations corresponding to Eq. (21). Joo et al. (1991a) used Fourier spectral methods with particular interest on the behavior of the first harmonics. The main goal was to investigate the problem of film rupture due to the thermocapillary effect (see Williams and Davis, 1982). They found that when Ma is large enough the perturbations grow in such a way that after some time the free surface touches the wall and the film rupture occurs. In case the heated film instability is three-dimensional, rupture may lead to the formation of rivulets, as shown by Joo et al. (1996). Kim (1999) solved numerically a nonlinear evolution equation of thin films under the effect of thermocapillarity and evaporation but with a wall thermal boundary condition of fixed heat flux. The time of rupture was investigated under this condition and compared with that of the fixed temperature gradient without and with evaporation.

A model equation including a capillary-induced interfacial evaporation term and another with streamwise thermocapillarity gave good numerical agreement with the experiments on film rupture done by Wang et al. (2000). Kliakhandler et al. (2002), by means of a Benney-type equation, investigate the flow of a condensate in one of the walls of a system of two vertical walls at different temperature. They find that vapor recoil and Marangoni effects are stabilizing in their system. Thiele and Knobloch (2004) investigate the relation and differences between the rupture patterns of a horizontal thin film and a slightly inclined one, both heated from below. In the horizontal case they found a large number of families of drop configurations which are stable due to the absence of drop interaction. However, for a very slight angle of inclination the drops interact due to a very thin film that appears between drops instead of the dry spots of the zero angle case. Ajaev (2004), in the lubrication approximation, investigates the simultaneous effects of themocapillarity and evaporation in a layer which is forming due to adsorption in a wall. The thickness of the layer is calculated by means of the disjoining pressure and thermodynamics. He finds that evaporation is stabilizing and has important effects in the velocity of the front of the capillary ridge. Tsvelodub (2008) calculates the stability of a thin film flowing down a vertical wall with temperature-dependent viscosity and thermal diffusivity. A nonlinear evolution equation is obtained in the small wavenumber approximation which is reduced to an equation similar to that of Kuramoto-Sivashinsky but with a term of different sign, which depends on the given relation between viscosity and thermal diffusivity on temperature. The equation is valid for large We. The solutions corresponding to the minus

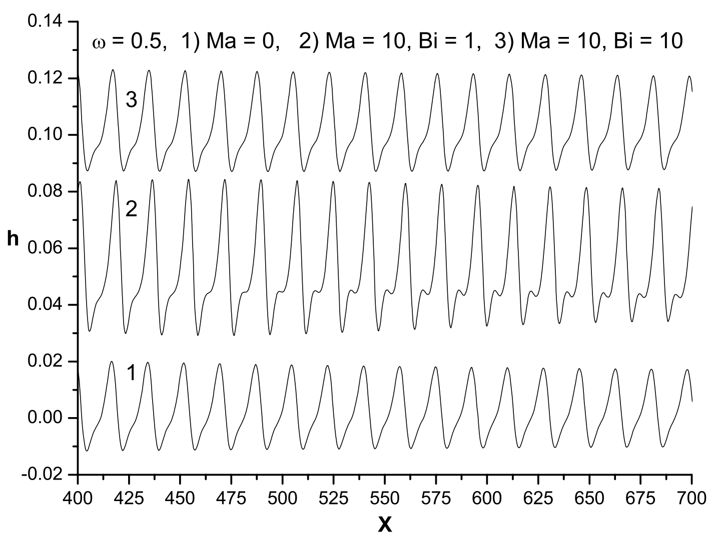

FIG. 8: $\beta = 90°$, $\omega = 0.5$, Re = 1.391 (maximum growth rate), time $T = 1000$. (1) Ma = 0 [see Eq. (21)], (2) Ma = 10, Bi = 1, (3) Ma = 10, Bi = 10. See Fig. 4 as reference to understand why the amplitude for Bi = 10 is smaller than that for Bi = 1.

sign decay and those of the plus sign are the same as those of the Kuramoto-Sivashinsky equation. Therefore, soliton solutions are expected.

4.2 The Ooshida equation

The limitation of the Benney type equations to order one Reynolds numbers and the lack of saturation below the curve of subcriticality (see Fig. 2 and Salamon et al., 1994) have motivated a number of researchers to find other equations which can be used in a broader range of magnitudes of Re and avoid singularity problems. This is the subject of this and the next subsections.

The discussion begins with the Ooshida (1999) model equation because it is an improved extension of the Benney equation model. Ooshida made a small wavenumber asymptotic expansion approximation of the variables and used the Padé approximants to improve and accelerate convergence of the series. The so-called regularized equation has the form

$$h_t - \frac{4}{21}\mathrm{Re}\left(h^5\right)_{xt} - \left(h^2 h_{xt}\right)_x + \frac{2}{3}\left[h^3 - \left(\frac{\cot\alpha}{4}h^4 + \frac{72}{245}\mathrm{Re}h^7\right)_x + \mathrm{We}h^3 h_{xxx}\right]_x = 0. \tag{24}$$

He tests Eq. (24) againts both full numerical analysis of the Navier-Stokes equations and experiments and finds a relation between the integral boundary layer equations and Eq. (24). He claims that the validity of the equation is restricted to $\mathrm{Re}/\mathrm{We}^{1/3} < 1$.

This model has been criticized by Scheid et al. (2005b) because the phase velocity and amplitude of the calculated solitary waves are underestimated at moderate Re. Ruyer-Quil et al. (2005) claim that the inertial terms of the Ooshida equation and that of Panga and Balakotaiah (2003) are equivalent and that therefore the last one has the same problems. The differences of this two models are presented numerically in Scheid et al. (2006) along with the differences of the new model presented in that paper.

Despite the problems of this model found by other researchers, the ideas presented in the paper by Ooshida (1999) have also been a source of inspiration in a number of publications. Examples are the papers by Panga et al. (2005) and Mudunuri and Balakotaiah (2006) where they already introduce the regularization method. They verified experimentally their analytical results in Meza and Balakotaiah (2008, 2011). Another example is the paper by Scheid et al. (2008) for a thermocapillary problem, where regularization is used too, but in a different way, in order to improve their previous models.

4.3 Integral boundary layer equations

Integral equations were proposed back to the works of Kapitza but it was formalized in the way used today by Shkadov (1967, 1968). The importance of the model is that the numerical results are valid for moderate Re up to 300 or less. This is not possible to attain with the Benney-type equations. Even though the results of the set of equations obtained by this method have some limitations, it is used widely in the literature.

The long wavelength approximation along with $\mathrm{Re} \sim 1/\varepsilon$ is assumed. The evolution equations are calculated first proposing an approximation of the main flow field in the direction of the flow down the wall. The form of a self-similar velocity profile was selected as a good choice when comparison was made with experimental results. The general form of that velocity is $u/U = f(\eta)$, where $\eta = z/h_1(x, y, t)$ is the self-similar variable (see Alekseenko et al., 1994) and U is a representative velocity proportional to the mass flux. In the case of two-dimensional flow, the equations are integrated and can be written in nondimensional form (Alekseenko et al., 1994; Shkadov, 1977) as

$$q_t + \frac{\beta_s}{\alpha_s^2}\left(\frac{q^2}{h}\right)_x = -\frac{\gamma_s}{\alpha_s}\frac{1}{\varepsilon\mathrm{Re}}\frac{q}{h^2} + \frac{3h}{\varepsilon\mathrm{Re}} - \frac{3\cot\beta}{\mathrm{Re}}hh_x + \frac{3\varepsilon^2\mathrm{We}}{\mathrm{Re}}hh_{xxx}, \tag{25}$$

$$h_t + q_x = 0, \tag{26}$$

where terms of higher order in ε have been neglected. The Eq. (26) comes from the kinematic boundary condition. Here $q = \int_0^h u dz$, $\alpha_s = \int_0^1 f(\eta)d\eta$, $\beta_s = \int_0^1 f(\eta)^2 d\eta$ and $\gamma_s = f(\eta)_\eta|_{\eta=0}$. The last one is related with the wall shear stress. Notice that q is a space and time function fluid flux because the integral is evaluated at $h(x, y, t)$.

In Eqs. (25) and (26) no particular approximation of the main velocity profile is used. They are general in this sense. Shkadov (2002) proposes that in case the system of equations is solved using Galerkin method the form of the velocity approximation is

$$u/U = \sum_{i=1}^{N} b_i(x, y, t)w_i(z). \tag{27}$$

With this, it could be possible to improve the approximation of the main velocity. However, the simplest parabolic profile $f(\eta) = 2\eta - \eta^2$ has shown to be very good in comparison with experimental results. In this case the momentum equation is

$$q_t + \frac{6}{5}\left(\frac{q^2}{h}\right)_x = -\frac{3}{\varepsilon \mathrm{Re}}\frac{q}{h^2} + \frac{3h}{\varepsilon \mathrm{Re}} - \frac{3\cot\beta}{\mathrm{Re}}hh_x + \frac{3\varepsilon^2 \mathrm{We}}{\mathrm{Re}}hh_{xxx}. \tag{28}$$

Notice that here Alekseenko et al. (1994) use Re = $gh_0^3 \sin\beta/3\nu^2$ and We = $\sigma/\rho g h_0^2 \sin\beta$ and not the definitions given above. Shkadov (2002) is able to reduce the number of nondimensional parameters to only one (Shkadov and Sisoev, 2004). He explains that the correlation with experimental results is good except for small magnitudes of a parameter of the form $\Gamma_k = \sigma/\rho(\nu^4 g)^{1/3}$, the Kapitza number. A comparison with the Orr-Sommerfeld solutions is done in Demekhin et al. (1988). With the change to a system moving with the phase velocity, it is possible to find stationary waves. It is shown that they form a large family (Bunov et al., 1984; Alekseenko et al., 1994; Bunov et al., 1984; Sisoev and Shkadov, 1999). The different families have been investigated extensively by Trifonov and Tsvelodub (1991), Tsvelodub and Trifonov (1992), Chang et al. (1993), Chang et al. (1995), Chang (1994) and Chang and Demekhin (2002). The series of wave bifurcations occurring in the solutions of this model is investigated numerically by Shkadov and Sisoev (2005). A comparison with experimental results is done by Tushkanov and Shkadov (2006) for the case of a slightly inclined plane. They show that the model reproduces correctly single-crested waves and solitary waves and all between this range of possibilities. The problem of wall topography was investigated by Trifonov (2007a,b). He found the possibility of stabilizing the flow by means of periodic wall deformations (see also Dávalos-Orozco, 2007, 2008). Sisoev (2008) investigates the limit cycles and wave bifurcation of the system of equations in order to explain the wavelength upper threshold found in experiments. Trifonov (2008) compares various models, including the present one, with full numerical analysis of the Navier-Stokes equations (Ramaswamy et al., 1996). He found that the full solution shows only few families of steady traveling waves but the integral method equations can have infinite different families. Heining et al. (2009) uses this model to investigate the space resonance which appears in the stability of a thin film flowing down a wavy wall.

The three-dimensional version was calculated by Demekhin and Shkadov (1984) for the first time. For thermocapillary effect these three-dimensional equations have been used by Kalliadasis et al. (2003a). The system has a local heat source. It is found that when the flow enters the heated plate a two-dimensional bump appears. They find an unstable band in the three-dimensional instability of the bump when the Marangoni number is increased. In contrast, in the work by Kalliadasis et al. (2003b) the heat is uniformly distributed in the wall. Their three-dimensional model equations are compared successfully against the linear numerical solutions of the Orr-Sommerfeld equation obtained by Goussis and Kelly (1991). They find solitary waves which agree with the solutions of the Benney equation when the Reynolds number is order one, but differ for larger magnitudes of Re . This three-dimensional version has also been used by Gambaryan-Roisman et al. (2011) and Sadiq et al. (2012) for thin films flowing down structured walls. The effect of evaporation has been investigated by Trifonov (1993). It is found that steady traveling waves appear and that film rupture is possible when the perturbation amplitude grows due to evaporation. This occurs after a critical film thickness is attained.

4.4 Weighted Residual Integral Boundary Layer Equations

The problems found in the linear solutions of the Benney equation and the integral boundary layer equations in relation with those of the Orr-Sommerfeld equation have been a motivation to investigate other approximations which may give more accurate models with respect to the linear full Navier-Stokes equations. However, those models also have to be in agreement with the results of the full numerical analysis of the Navier-Stokes equations in a wide range of the parameters.

Another attempt has been made by Ruyer-Quil and Manneville (1998), combining the small wavenumber expansion with a weighted residuals method used regularly to solve boundary value problems of differential equations (Finlayson, 1972). The method was improved by Ruyer-Quil and Manneville (2000, 2002) to adjust even more to numerical and experimental results. Their first order in ε, optimized system of equations (Ruyer-Quil and Manneville, 2002) are formed by Eq. (26) and an equation, which in their nondimensional form is

$$q_t = \frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} - \frac{17}{7}\frac{q}{h}q_x + \left(\frac{9}{7}\frac{q^2}{h^2} - \frac{5}{6}h\cot\beta\right)h_x + \frac{5}{6}\mathrm{We}\,(3\mathrm{Re})^{2/3}\,hh_{xxx}. \tag{29}$$

The second order optimized system of equations is more complex. It is found that the linear relaxation times of the two new variables are shorter than those of q. Therefore, they make an adiabatic elimination which is used to obtain the approximated system formed by Eq. (26) and by

$$q_t = \frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} - \frac{17}{7}\frac{q}{h}q_x + \left(\frac{9}{7}\frac{q^2}{h^2} - \frac{5}{6}h\cot\beta\right)h_x + 4\frac{q}{h^2}(h_x)^2 - \frac{9}{2h}q_x h_x - 6\frac{q}{h}h_{xx} + \frac{9}{2}q_{xx} + \frac{5}{6}\mathrm{We}\,(3\mathrm{Re})^{2/3}\,hh_{xxx}. \tag{30}$$

The linear solution of these equations has no problem for small Re as had the integral boundary layer Eq. (28) with respect to the Orr-Sommerfeld equation. Moreover, they show that these equations agree in a large range of Γ_k with full numerical solutions of the Navier-Stokes equation by Ramaswamy et al. (1996) and with experiments by Liu and Gollub (1994).

This problem is extended to comprise thermocapillary effects by Trevelyan and Kalliadasis (2004) including a high-order Galerkin projection of the heat diffusion equation. They compare successfully their linear results with the full linear equations. From the nonlinear point of view, they find the existence of solitary waves for any Re with no blow-up behavior.

Following this improvement over other equations' results, an important step is to delimit the range of validity of the equations obtained by this method and, at the same time, set the restrictions of the parameter range for the other well-known equations. One step has been given by Scheid et al. (2005a), who examine the magnitudes of the parameters where a Benney-type equation with thermocapillary effects blows up. Therefore, they compare the traveling wave solution with those of a weighted integral boundary layer equation. The system of equations is calculated with the fixed temperature at the wall and the Newton's cooling law at the free surface [Eq. (11)]. It is formed by Eq. (26) coupled with the following equation:

$$q_t = \frac{5}{6}\sin\beta h - \frac{5}{2}\frac{q}{h^2} - \frac{17}{7}\frac{q}{h}q_x + \left(\frac{9}{7}\frac{q^2}{h^2} - \frac{5}{6}h\cos\beta\right)h_x + \frac{5}{6}\mathrm{Ka}hh_{xxx} + \frac{5}{4}\frac{\overline{\mathrm{Ma}}\overline{\mathrm{Bi}}}{\left(1+\overline{\mathrm{Bi}}h\right)^2}h_x, \tag{31}$$

where Ka = $3\Sigma\nu^{2/3}/h_0 g^{1/3}$ is the Kapitza number, $\overline{\mathrm{Ma}} = \mathrm{Ma}\nu^{2/3}/\,\mathrm{Pr}\,h_0 g^{1/3}$, and $\overline{\mathrm{Bi}} = \mathrm{Bi}\nu^{2/3}/h_0 g^{1/3}$. One finding is that the one hump solitary waves are the first to blow up except when the Reynolds number is smaller than a calculated value. It is found that the stationary waves are bounded when the wavenumber is small enough. They also compare successfully in the strong nonlinear regime with the full numerical analysis of Ramaswamy et al. (1996).

The equations for thermocapillarity have been improved using a gradient expansion along with a Galerkin projection by Ruyer-Quil et al. (2005). They fix the temperature at the wall to calculate a coupled set of equations which includes Eq. (26) and an extra equation for temperature θ. Thus, to first order the other two equations are:

$$q_t = \frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} - \frac{17}{7}\frac{q}{h}q_x + \left(\frac{9}{7}\frac{q^2}{h^2} - \frac{5}{6}h\cot\beta\right)h_x + \frac{5}{6}\overline{\Gamma}hh_{xxx} - \frac{5}{4}\overline{\mathrm{Ma}}\theta_x \tag{32}$$

$$\mathrm{Pr}\,\theta_t = \frac{3}{h^2}\left[1 - \left(1+\overline{\mathrm{Bi}}h\right)\theta\right] + \mathrm{Pr}\left[\frac{7}{40h}(1-\theta)\,q_x - \frac{27q}{20h}\theta_x\right], \tag{33}$$

where $\overline{\Gamma} = 3\Sigma\nu^{2/3}/h_0(g\sin\beta)^{1/3}$ is the Kapitza number, $\overline{\mathrm{Ma}} = \mathrm{Ma}\nu^{2/3}/\,\mathrm{Pr}\,h_0(g\sin\beta)^{1/3}$, and $\overline{\mathrm{Bi}} = \mathrm{Bi}\nu^{2/3}/h_0\times(g\sin\beta)^{1/3}$. They also calculate equations upto the second order. These equations have been used to compare with the Benney equation and the integral boundary layer equation to set their range of validity by Scheid et al. (2005b). This time they propose a regularized reduced model which agrees very well with the Orr-Sommerfeld equation and it is found that the nonlinear one-humped solitary wave solutions exist for all Re.

The Padé approximant technique proposed by Ooshida (1999) is used in Scheid et al. (2006) in another way to obtain new equations which are compared with numerical analysis and experimental results. The new regularized model equations are

$$q_t = \frac{9}{7}\frac{q^2}{h^2}h_x - \frac{17}{7}\frac{q}{h}q_x + \frac{\mathrm{We}^{1/3}}{3\mathrm{Re}}\frac{5}{6}h - \frac{5}{2}\frac{q}{h^2} + \frac{1}{\mathrm{We}^{2/3}}\left[4\frac{q}{h^2}(h_x)^2 - \frac{9}{2h}q_x h_x - 6\frac{q}{h}h_{xx} + \frac{9}{2}q_{xx}\right] - \frac{5}{6}\frac{\cot\beta}{\mathrm{We}^{1/3}}hh_x + \frac{5}{6}hh_{xxx}\left[1 - \frac{3\mathrm{Re}}{70\mathrm{We}^{1/3}}qh_x\right]^{-1} \tag{34}$$

coupled to Eq. (26). For validation, they compare with full numerical analysis of Salamon et al. (1994), Ramaswamy et al. (1996), and Malamataris et al. (2002). In this paper the three-dimensional instability is investigated for the first time using the approximation under discussion.

The model is applied in Trevelyan et al. (2007) using a high-order Galerkin projection in a problem which includes large Péclet number and Biot number of the wall. The three-dimensional instability is extended to include thermal Marangoni effects in Scheid et al. (2008). They find that in the small Reynolds number flow rivulets form in the direction of the flow down the wall where no waves are present. For larger Re, rivulets do not appear and waves are observed as in the isothermal case. They make a map of the regimes found in their calculations.

Oron et al. (2009) investigate deeply the first-order model and find numerically the existence of traveling waves and aperiodic nonstationary waves. Moreover, it is shown that they may coexist. They confirm that bounded solutions exist for large Re under periodic boundary conditions. However, it is observed that the solutions can show a reverse flow (negative q) in a direction contrary to gravity. They compare with full numerical analysis of Ramaswamy et al. (1996) and Nosoko and Miyara (2004) to set the range of validity.

This model has been extended to include a countercurrent gas flow by Tseluiko and Kalliadasis (2011). The application to curved walls has been investigated by Ruyer-Quil et al. (2008) and Ruyer-Quil and Kalliadasis (2012) in the case of flow down a fiber.

5. NON-UNIFORM HEATING

This section reviews thin films under nonuniform heating. Only three kinds of non-uniformities are discussed. One is due to local heating, that is, when the falling film abruptly meets with a hot spot in the wall. The idea is to understand the possibility of cooling electronic devices, for example. Next, a discussion is given on the influence of a sinusoidal temperature and linear longitudinal temperature increase along the wall in the flow direction. Assuming that the atmosphere has a constant temperature, these produce an increase of the destabilizing temperature gradient along the flow direction. Finally, a survey of flow down a wavy wall is presented. It is shown that, even though the wall might be heated uniformly, the main temperature solution, calculated under the lubrication approximation, depends on the waviness of the wall which introduces an effective variation in the direction of the main flow down the wall.

5.1 Local Heating

The problem of local heating was investigated by Kabov et al. (2001) experimentally and numerically. They find that a heat increase produces an instability which leads to the formation of horseshoe patterns. Then they observe the appearance of streamlets with increase of heat. It is observed that the film thickness decreases with distance. The agreement with numerical analysis was satisfactory only when both temperature-dependent surface tension and viscosity were included into the model. Kabov et al. (2002) investigate the influence of the heater on the shape of the free surface wave and the formation of rivulets. They find that the bump shape is unstable when the heat flux is increased. Above a critical heat flux, it is observed in the horseshoe patterns a counterflow due to thermocapillarity. Chinnov and Kabov (2003) investigate the formation of jets and the distance separating them. They find that there is a thermocapillary-wave mechanism apart from that of the formation of jets. They change the distance of the heater from the nozzle to investigate the behavior of different waves in the heater. If two-dimensional waves arrive they become three-dimensional in the heater and, as the heat flux is increased, the distance between wave crests is reduced. They find that it is possible that the jet spacing has the influence of the nonuniform distribution of heat at the wall.

Skotheim et al. (2003) investigate numerically the linear stability of a long wave model and look for the critical Marangoni number in the presence of a heater. They find that the stresses due to the Marangoni effect can oppose the flow due to gravity and form a ridge. They conclude that the instability leading to the formation of the ridge is different from the one that drives contact lines. Kalliadasis et al. (2003a) attack this problem by means of the integral boundary layer equations in three dimensions. They find a bump at the heated plate and investigate its stability. It is found that above a critical Marangoni number a discrete mode is unstable, but far from this critical one there is a band of unstable discrete modes. Full numerical analysis has been done by Frank (2003) of a flow down a wall with a rectangular plate. He is able to reproduce successfully experimental wave profiles, the bump, and rivulets.

Chinnov et al. (2004) use a dielectric liquid to investigate the effect of a heater for a wide range of Reynolds numbers. The effect of the heat flux in the symmetry line of the heater is investigated. Enhanced heat transfer is found for small Reynolds numbers under intense evaporation and when rivulets start to appear. They also characterize the regions of continuous film and rivulets. Zaitsev and Kabov (2005) investigate the three-dimensional instability occurring in the heater with a noncontact fiber optical probe. They find that the decrease of wave amplitude between the rivulets is due to the reduction of the local Reynolds number (the film is thinner between rivulets) and is not related to thermocapillary effects. Frank and Kabov (2006) are interested in the pattern formation inside the heater and make both experiment and numerical analysis. They find the critical Marangoni number depending on the Reynolds and Weber number along with the wavenumber of the maximum growth rate. It is found that the height of the wave bump at the entrace of the heater depends on the inverse of the Reynolds number. Besides, when the reverse flow appears in the hump, not necessarily instability does occur.

Zaitsev et al. (2007) investigate the conditions for the rupture of the thin film when passing on the heater. Grooves appear in the heater and it is found that the heat flux needed to produce stable dry patches is twice as high as that of a smooth heater. An extra effect is that the dry patches cannot spread in the heater plate due to the grooves. Kabova et al. (2007) use numerical analysis to investigate the three-dimensional patterns which appear in the heater. They find that there exists an interaction between the hump and the lateral instabilities. It is shown that there is a relation between lateral waves and the film thickness. A review of these phenomena upto 2010 has been published by Kabov (2010).

Tiwari et al. (2007) investigate the instability of the bump against spanwise perturbations by means of the lubrication approximation. They found a finite band of wavenumbers for instability after a critical Marangoni number. These instabilities lead to the formation of rivulets. It is shown that the wavelength of the instability increases as the angle of inclination decreases. Tiwari and Davis (2009a) include the effect of evaporation when investigating the stability of the film. In this case, the critical Marangoni number has a nonmonotonic dependence on the steepness of the heater temperature gradient, in contrast to the problem investigated before by Tiwari et al. (2007). It is found that the stability is more sensitive to the temperature gradient and heater width than to the influence of the other parameters of the problem. This is extended to a nonmodal analysis of the problem with evaporation by Tiwari and Davis (2009b). They found a Hopf bifurcation increasing the Marangoni number from criticality, which in the linear case destabilizes into a transverse perturbation. Tiwari and Davis (2010) introduce the effect of wall deformations in the problem of local heating. It is of interest because in isothermal flow a film passing through a small step forms a stable ridge above the highest part of the step, in contrast to the unstable one in the heated finite plate. Therefore, their goal is to find the adequate topography which allows suppression of the transverse instabilities.

Chinnov and Shatskii (2010) use a high-speed infrared imager to investigate the transformation of hydrodynamic instabilities into thermocapillary wave instabilities. They found that the inhomogeneous temperature profile formed in the hydrodynamic wave is responsible for the instability leading to rivulets.

Tiwari and Davis (2011) are interested in the regime where evaporation is in balance with the gravity flow. For small Ma the flow is steady but for large magnitudes the flow becomes a periodic one which destabilizes into rivulets. It is found that the length of the fingers is limited by evaporation and the time periodic profiles may lead to oscillating rivulets.

Chinnov et al. (2012a) investigate experimentally the change of three-dimensional patterns into thermocapillary wave patterns. Use is made of a high-speed infrared recording technique to measure the temperature. It is found that the temperature perturbations occur is a residual layer of the film and that temperature inhomogeneities occur in the three-dimensional wave front which are the thermocapillary cause of rivulets. Chinnov (2012b) investigates experimentally this problem for the first time for Re = 150. It is found that the patterns have temperature maxima that are more near in space than in the other conditions investigated before. Kabova et al. (2012) perform numerical analysis of thin films which have temperature-dependent viscosity. It is supposed that the viscosity and surface tension vary linearly with temperature. They obtain a general formula for the film thickness in terms of the flow rate, among other parameters. Another deformation upstream is observed besides the usual bump found in other investigations. This new deformation does not appear if the angle of inclination is below a critical value.

5.2 Linear and Sinusoidal Variation of Temperature Down the Wall

The problem of a thin film flowing down a wall which has a temperature increasing in the direction of the flow was first investigated by Miladinova et al. (2002a). They calculate an evolution equation of the Benney type in the longwave approximation. They manage to avoid the explicit appearance of the x coordinate in the evolution equation. One important assumption is that Ma is very large and is scaled in a way that it appears to be influencing the phase velocity. From the linear stability analysis it is found that an increase of wall temperature reduces the critical Re and its decrease stabilizes the flow. The magnitude of the phase velocity depends on the sign of the temperature gradient on the wall. They make a weakly nonlinear theory of the evolution equation and find the curve of subcriticality. Inside this theory they predict a supercritical bifurcation and the presence of solitary waves. Miladinova et al. (2002b) investigated the two-dimensional waves instability into three-dimensional ones and found that they are sensitive to the longitudinal temperature gradient. When the temperature decreases along the wall the film stabilizes and the 3D instability depends on the Reynolds number, but if the temperature increases it depends on the Marangoni number.

An extension of the problem has been done by Sadiq and Usha (2005) including viscoelastic effects of a Walters fluid. They investigated the linear instability and found for this fluid model that the heated viscoelastic film is more stable than the isothermal Newtonian film, The contrary occurs when the film is cooled.

The goal of Demekhin et al. (2006) is to suppress the instability of the falling film by this horizontal temperature gradient. The linear instability is used to find the existence of long surface waves, convective rolls, and longitudinal convective rolls. Convective rolls can appear for long and short wave instabilities. They found that for large Pr, an increase in temperature gradient first destabilizes and then stabilizes and, for small Pr, stabilizes.

Mukhopadhyay and Mukhopadhyay (2007) present a nonlinear evolution equation and make a multiple scales expansion to investigate the linear and nonlinear solutions. It is found that the phase velocity is nondispersive in the linear case, but not in the nonlinear one. It is shown that the supercritical and subcritical regions are influenced by Ma in such a way that, for example, in the linear supercritical one the growth rate is positive but in the nonlinear one the growth rate is negative. This research is complemented for the nonlinear problem by Samanta (2008), who calculates a nonlinear evolution equation and uses multiple scales to obtain a complex Ginzburg-Landau equation to understand the stability of traveling waves. He finds that an increase of the product Re Pr decreases the subcritical stable region and increases the supercritical unstable region. Mukhopadhyay and Mukhopadhyay (2011) extend the problem of Dávalos-Orozco and Busse (2002) for a film flowing down a rotating inclined plane to include the effects of nonuniform heating of the wall. They calculate an evolution equation of the Benney-type including small rotation rates. They find that the destabilizing effect of rotation is increased with Ma and that the phase velocity first decreases with rotation and then increases after a critical magnitude. Nonlinear stability calculations are also presented to describe the behavior of the sub and supercritical regions.

Scheid et al. (2002) investigate the thin film response to a sinusoidally heated wall and compare results with the case of uniformly heated wall. It is found that for moderate heating the waves are modulated. But, when the temperature increases no oscillatory traveling waves are detected, only steady deformations. One or the other free surface behaviors has important influence on the heat transfer coefficient. Miladinova and Lebon (2005) add the effect of evaporation to the wavy heating of the wall. From the numerical analysis of the Benney-type equation, multiple humps are observed before the film rupture occurs due to mass loss. The film rupture appears near the midpoint of the maximum and minimum temperatures of the wall. The evaporation of the film decreases the amplitude of the perturbation promoting its damping.

5.3 Effect of Wall Waviness

Research on the effect of deformations of the wall on the film instability has been of interest for many years. In particular, the effect of periodic waviness of the wall has been investigated by many researchers due to its destabilizing and stabilizing effects under different conditions of wall geometry and flow. Experimental and theoretical investigations have been done by Bontozoglou and Papapolymerou (1997), Malamataris and Bontozoglou (1999), Vlachogiannis and Bontozoglou (2002), Wierschem et al. (2002), Wierschem and Aksel (2003), Scholle et al. (2004), Trifonov (1998, 2004, 2007a,b), Heining et al. (2009), Heining and Aksel (2010), Wierschem et al. (2010), and Nguyen and Bontozoglou (2011).

Heining et al. (2009) find a process from which it is possible to reconstruct the unknown bottom topography based on the observed free surface deformation and Heining (2011) is able to reconstruct the velocity field when the wall deformations are unknown. Pascal and D'Alessio (2010) include the effect of a porous wavy wall. Pak and Hu (2011) and Veremieiev et al. (2012) investigate the effect of an electric field. Oron and Heining (2008) use a weighted-residual integral boundary layer model equation and Hacker and Uecker (2009) obtain an integral boundary layer equation to investigate this problem. The mixing due to the flow field and the wall corrugation is investigated by Heining et al. (2012). For the effect of three-dimensional topographical obstacles, see the papers by Veremieiev et al. (2010, 2011, 2012), Baxter et al. (2009), and Blyth and Pozrikidis (2006).

Dávalos-Orozco (2007) obtained a Benney-type equation including a function defining the wall deformation. In the case of a wavy wall it is found numerically that time-dependent perturbations imposed on the free surface can be stabilized when it happens that the wall wavelength and the time-dependent perturbation wavelength have a particular relation which leads to a spatial resonance. Under this resonance the film response to the wall deformation produces large free surface deformations which decrease in great measure the film thickness in some regions and increase similarly in others. In regions where the film is very thin the local Re is very small and the time-dependent perturbations decrease (stabilize) in such a way that they are able to fade away in space and time. The problem when the wall waviness has a finite extension was investigated by Dávalos-Orozco (2008). It is shown that if the resonance occurs fast enough, even in a finite space interval, it is possible to stabilize in a passive way the time-dependent perturbations.

Now the problem is to find out if this resonant effect is also possible in case the liquid layer is heated from below and thermocapillary effects are present. It can be shown that, following the same lubrication approximation for the temperature (see for example Joo et al., 1992; Dávalos-Orozco, 2012), the main temperature profile has the form

$$T_0 = \mathrm{Bi}\frac{\zeta - z}{1 + \mathrm{Bi}h}, \tag{35}$$

where $\zeta(x, y)$ is the deformation of the wall. Notice that when the wall is deformed the boundary conditions at the wall Eqs. (6) and (19) are now evaluated at $z = \zeta(x, y)$ and those at the free surface Eqs. (7) to (9), and (11) are evaluated at $z = \zeta(x, y) + h(x, y, t)$ (see Dávalos-Orozco, 2007).

In the lubrication approximation the temperature T_0 is enough to describe the thermocapillary effects. Therefore, the presence of $\zeta(x, y)$ in Eq. (35) shows that the heating effect of a wavy wall is nonuniform under this approximation. However, surprisingly, when the temperature is fixed at the wall, this $\zeta(x, y)$ cancels cleanly and does not appear in the term including the Marangoni number, as can be seen in the following Benney-type evolution equation:

$$\begin{aligned} h_t + \mathrm{Re}\sin\beta h^2 h_x + \varepsilon\Bigg\{ (\mathrm{Re}\sin\beta)^2 \left(\frac{2}{15}h^6 h_x\right)_x + \frac{1}{3}\nabla\cdot\Bigg[h^3\left(-\mathrm{Re}\cos\beta\nabla(\zeta+h) + 3S\nabla^2\nabla(\zeta+h) - \nabla P_p\right) \\ + \frac{1}{2}\frac{\mathrm{Ma}}{\mathrm{Pr}}\frac{\mathrm{Bi}h^2}{(1+\mathrm{Bi}h)^2}\nabla h\Bigg]\Bigg\} = 0. \end{aligned} \tag{36}$$

It is clear when $\zeta = 0$, Eq. (36) reduces to Eq. (32). Therefore, in the lubrication approximation, valid for order one Reynolds numbers, the wall topography effects only appear in the same terms as in the isothermal equation obtained in

Dávalos-Orozco (2007, 2008). Under these conditions, it might be difficult to stabilize time-dependent perturbations in the presence of thermocapillary effects. However, for some magnitudes of the Marangoni, Prandtl and Biot numbers it is still possible to find a resonance wall wavenumber such that stabilization is attained. This is shown by means of the numerical results of Eq. (36) calculated by the author and shown in the following figures. Notice that this Eq. (36) has been presented in Appendix A of D'Alessio et al. (2010). However, this is the first time that numerical results appear in the open literature. The curves are calculated using the Reynolds number of the maximum growth rate corresponding to the given frequency ω of the time-dependent perturbations (see Fig. 3 and Fig. 4). All the results shown in the figures are calculated for $\beta = 90°$, Pr = 7, and $S = 1$. The wavenumber of the wall is determined as k/L. Then, the wavelength of the wall is L times that of the time dependent perturbations. In each case, spatial resonance occurs for a particular magnitude of L (see Dávalos-Orozco, 2007, 2008).

The first set of Figs. 9 and 10 are for Bi = 10. However, note that the results for Bi = 10 are the same as for Bi = 0.1, as shown in Fig. 4. Figure 4 also shows that the second set of Figs. 11 and 12 for Bi = 1, are results where thermocapillarity shows its maximum strength. Each figure reviews the numerical results presenting two sets of three curves each. For the sake of comparison, the set of curves given above are the results for a film flowing down a flat wall. As can be seen in Figs. 3 and 6, when Re has the magnitude of the maximum growth rate for the given frequency, it is easier for a perturbation to come inside the subcritical region when the frequency is small than when it is large. In that former case the Re is small and in the latter case Re is large. This is a result of the stabilizing effect of the Reynolds number in some regions of the (k, Re) plane in the presence of thermocapillarity.

The results in Fig. 9 are for a frequency ω = 0.5 and Re = 1.391. Here, for L = 5, it is difficult to stabilize by spatial resonance the waves when Ma = 100. In contrast, in Fig. 10 for ω = 2.5 and Re = 3.111 it is possible to stabilize with L = 8 for all the Ma investigated. When Bi = 1, ω = 0.5, and Re = 1.391, the instability is below subcriticality. Then, Fig. 11 shows that it not possible to stabilize for Ma = 50 even though L is reduced to 4.4. The case Ma = 100 (far below subcriticality) is not presented due to lack of convergence of the numerical algorithm. The last Fig. 12 has parameters Bi = 1, ω = 2.5, and Re = 3.111. The curve for Ma = 100 can still be stabilized in space a little in comparison with the flat wall case. It is concluded that under this approximation there are magnitudes of Ma above which the film time-dependent perturbations cannot be stabilized, depending also on the other parameters of the problem.

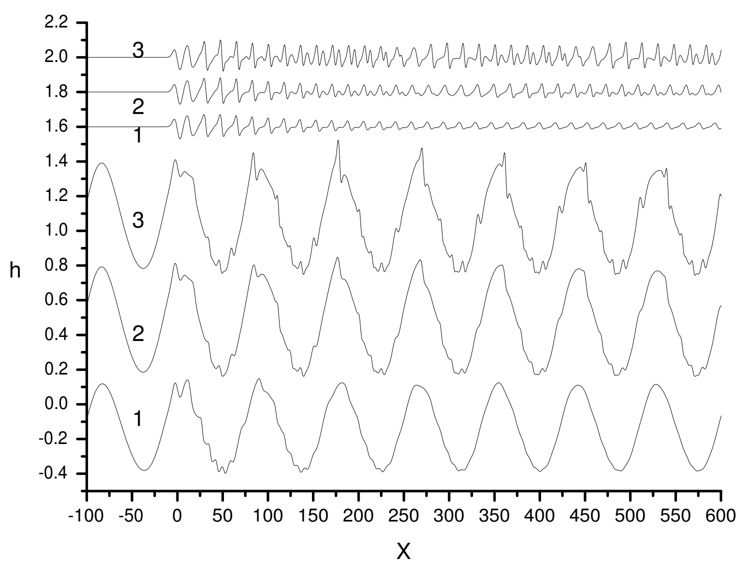

FIG. 9: Heated wavy wall. $\beta = 90°$, Pr = 7, S = 1. Review of results for ω = 0.5, Re = 1.391, Bi = 10 and T = 1000. All the curves are for L = 5 and correspond to (1) Ma = 10, (2) Ma = 50, and (3) Ma = 100, respectively. The three curves shown above are their corresponding flows down a flat wall.

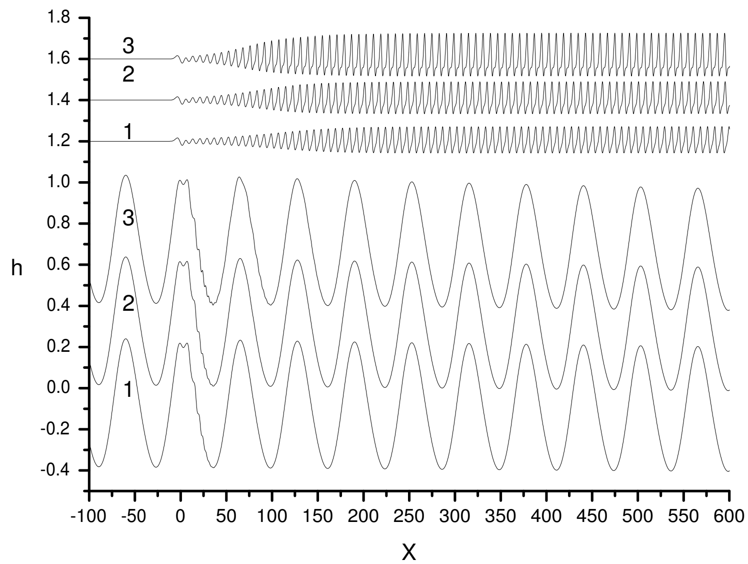

FIG. 10: Heated wavy wall. $\beta = 90°$, Pr = 7, S = 1. Review of results for ω = 2.5, Re = 3.111, Bi = 10, and T = 600. The curves are for L = 8 and correspond to (1) Ma = 10, (2) Ma = 50, and (3) Ma = 100, respectively. The three curves shown above are their corresponding flows down a flat wall.

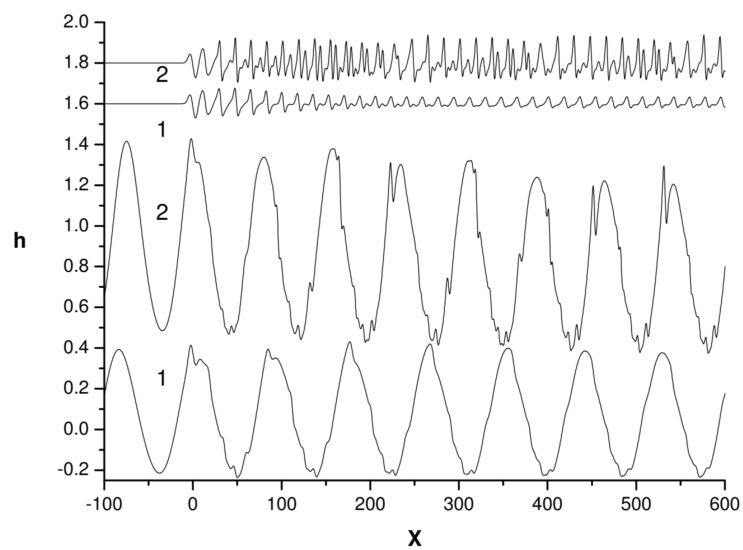

FIG. 11: Heated wavy wall. $\beta = 90°$, Pr = 7, S = 1. Review of results for ω = 0.5, Re = 1.391, Bi = 1, and T = 1000. The curves are for (1) Ma = 10 and L = 5, and for (2) Ma = 50 and L = 4.4, respectively. The two curves shown above are their corresponding flows down a flat wall.

From the above results, the need is clear to investigate the problem when the effects of inertia are important in order to widen the range of the Reynolds number far above order one. D'Alessio et al. (2010) made calculations of the weighted-residual integral boundary layer model to attack the thermocapillary problem, following their paper

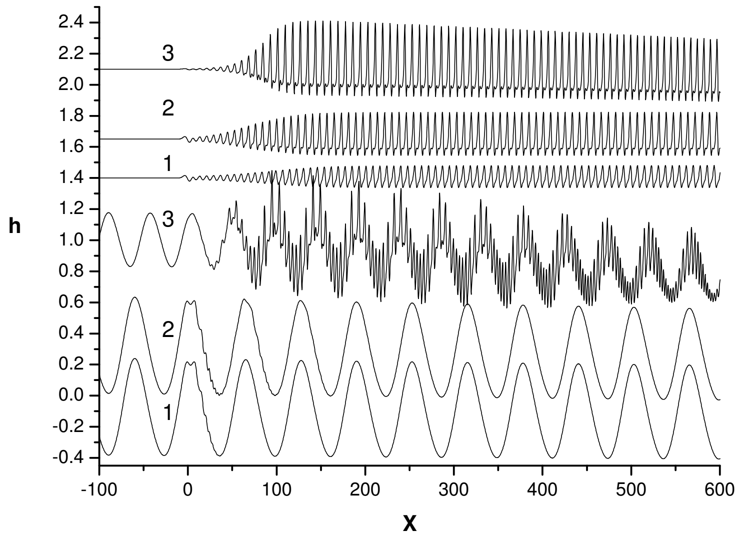

FIG. 12: Heated wavy wall. β = 90°, Pr = 7, S = 1. Review of results for ω = 2.5, Re = 3.111, Bi = 1, and T = 600. The curves are for (1) Ma = 10 and L = 8, (2) Ma = 50, and L = 8 and for (3) Ma = 100 and L = 6, respectively. The three curves shown above are their corresponding flows down a flat wall.

(D'Alessio et al., 2009). Their calculations lead to a set of equations of second order which couple the mass flux, film height, and temperature. The equations coupled to Eq. (26) are:

$$q_t + \left[\frac{9}{7}\frac{q^2}{h} + \frac{5}{4}\frac{\cot\beta}{\mathrm{Re}}h^2 + \frac{5}{4}\mathrm{Ma}\theta\right]_x = \frac{9}{7h}q_x + \frac{5}{2\varepsilon\mathrm{Re}}\left(h - \frac{q}{h^2}\right) + \frac{5}{6}\varepsilon^2\mathrm{We}h\left(h_{xxx} + \zeta'''\right) - \frac{5\cot\beta}{2\mathrm{Re}}\zeta' h$$
$$+ \frac{\varepsilon}{\mathrm{Re}}\left[\frac{9}{2}q_{xx} - \frac{9}{2h}q_x h_x + \frac{4q}{h^2}\left(h_x\right)^2 - \frac{6q}{h}h_{xx} - \frac{5\zeta' q}{2h^2}h_x - \frac{15\zeta'' q}{4h} - \frac{5\left(\zeta'\right)^2 q}{h^2}\right]$$
$$+ \frac{\varepsilon\mathrm{ReMa}}{16}\left[\frac{h^2}{3}\theta_{xt} + \frac{15hq}{14}\theta_{xx} + \frac{19h}{21}q_x\theta_x + \frac{5q}{7}h_x\theta_x\right] \tag{37}$$

$$h\theta_t + \frac{27q}{20}\theta_x - \frac{7}{40}\left(1-\theta\right)q_x = \frac{3}{\varepsilon\mathrm{Re}\,\mathrm{Pr}\,h}\left[1 - \theta\left(1 + \mathrm{Bi}h\right)\right] + \frac{\varepsilon}{\mathrm{Re}\,\mathrm{Pr}}\left\{\left(1-\theta\right)h_{xx} + h\theta_{xx} + h_x\theta_x\right.$$
$$\left. - \left[\frac{3\mathrm{Bi}\theta}{2} + \frac{2(1-\theta)}{h}\right]\left(h_x\right)^2 - \frac{3\zeta'}{h}\left(1 - \theta + h\mathrm{Bi}\theta\right)h_x + \frac{3\zeta''}{2}\left(1-\theta\right) - \frac{3\mathrm{Bi}(\zeta')^2\theta}{2}\right\}$$
$$+ \frac{3\varepsilon\mathrm{Re\,Ma}}{80}\left[2h^2\left(\theta_x\right)^2 - h^2\left(1-\theta\right)\theta_{xx} - 2h\left(1-\theta\right)h_x\theta_x\right]. \tag{38}$$

Here, the dashes mean total derivative of the wall profile. From Eqs. (37) and (38) it is clear how in a large Reynolds number approximation the wall profile also has influence on the film stability through the thermocapillary terms of the coupled set of equations.

D'Alessio et al. (2010) show that the linear stability influence of We when the wall is wavy is very different in comparison to the lubrication approximation. In a range, We stabilizes but above a magnitude of We destabilization occurs. However, Ma remains as a destabilizing parameter. It is shown that the effect of We also depends on the inclination angle. The bottom amplitude is important in the stability associated with We. For large We, an increase of the amplitude leads the critical Re to a maximum after which it decreases, destabilizing the flow. For some values of the parameters Ma can stabilize a flow for large We in comparison to the critical Re of smaller We. They make full numerical analysis and compare with both experiment and with the results of their large Re approximate equations. The agreement they found is satisfactory.

The case when the wall is heated and its waviness is aligned with the flow direction down the wall was investigated by Gambarian-Roisman and Stephan (2009). Their results, which include the effects of disjoining pressure, show the appearance of rivulets. Ogden et al. (2011) investigate the effect of heat transfer, thermocapillarity, and the presence of a porous wall on the stability of the film flow. They use the method of weighted-residuals integral boundary layer method to calculate a set of evolution equations.

6. EFFECT OF THICKNESS AND THERMAL CONDUCTIVITY OF THE WALL

It is a common practice to obtain model equations of thin films flowing down heated walls without taking into account the thickness and heat conductivity of the wall. The usual thermal boundary condition applied at the interface between the liquid and the wall is that of fixed temperature. That is, the perturbation of the temperature is zero at that interface. It is supposed that this condition applies, in approximation, when the wall heat conductivity is very high. However, it is also possible to obtain this approximation when the thickness of the wall is used. To this goal, the solution of the heat diffusion equation of the wall has to include the location of the lower limit of the wall and the location of the interface between the wall and the liquid film. In this way, the wall and the liquid are thermally coupled. In this process, the thermal conductivities come into play when the condition of continuity of heat flux is enforced at the wall-fluid interface.

Some researchers have used these two properties but their goal was not directly to understand their effect on the stability of the film. Oron et al. (1996) included the thickness of the wall to avoid the appearance of singularities in the temperature, heat, and mass flux in a problem of breakup of an evaporating film. The singularities are due to the existence of two different temperatures at two locations which can merge. With the wall thickness they were able to obtain a nonlinear evolution equation to describe the instability of an evaporating film acceptable under different limit magnitudes of the parameters. Kabova et al. (2006) obtained an evolution equation in the small wavenumber approximation and made full numerical calculations of the instability of a horizontal thin film laying on a wavy wall heated from below. They included the thick wall because of the resemblance to experimental settings and also to introduce the wall deformations. However, they do not use the thickness and conductivity of the wall as parameters of their problem. They were mainly interested in the possibility of the local rupture of the film due to the wall waviness. They found that when the film is heated the fluid tends to accumulate at the deep parts of the wavy deformations. When Ma is not too large, a steady state is attained where the characteristic is the appearance of vortices. They conclude that the films on wavy walls are more unstable than those on flat walls.

Gambarian-Roisman and Stephan (2009) also used the thickness of the wall to allow for the introduction of wall waviness at the interface between the wall and the fluid. However, they do not investigate the effect of thickness variations. Gambarian-Roisman (2010) also includes the thickness of the wall but her research is focused mainly on the nonuniform variation of the wall heat conductivity. She uses the thickness as a reference length for the thermal conductivity variation. The results show that in fact the variation of the thermal conductivity of the wall has an important influence on the free surface thermocapillary perturbations. Moreover, it is shown that a critical magnitude of the thermal conductivity variation exists which induces the possibility of film rupture.

Recently, a more systematic research has been done by Dávalos-Orozco (2012) on the effect of the wall thickness and thermal conductivity on the Marangoni instability. In the small wavenumber approximation an evolution equation has been obtained which includes a parameter which contains the thicknesses ratio and heat conductivities ratio. The equation is

$$h_t + \mathrm{Re}\sin\beta h^2 h_x + \varepsilon\left\{(\mathrm{Re}\sin\beta)^2\left(\frac{2}{15}h^6 h_x\right)_x + \frac{1}{3}\nabla\cdot\left[h^3\left(-\mathrm{Re}\cos\beta\nabla h + 3S\nabla^2\nabla h - \nabla P_p\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\frac{\mathrm{Ma}}{\mathrm{Pr}}\frac{\mathrm{Bi}h^2}{\left[1+\mathrm{Bi}h+\mathrm{Bi}\left(d/Q_c\right)\right]^2}\nabla h\right]\right\} = 0. \tag{39}$$

The linear stability results have already been plotted in Figs. 5 and 6. The plot of Fig. 5 shows the important effect the parameter d/Q_c has on the coefficient of the thermocapillary term of the evolution equation. When d/Q_c is large the thermal effects are small and when it is zero the usual thermocapillary effects are present depending on Bi. This is reflected in the curves of criticality and subcriticality, as shown in Fig. 6. A sample of numerical solutions of Eq. (39) is given in Figs. 13 and 14. These nonlinear results correspond to the linear ones of Fig. 6, which were calculated for a film falling down a vertical wall, Bi = 0.1 and Ma = 50. The Reynolds numbers used correspond to those of the maximum growth rate of the isothermal problem which is used as reference in the nonlinear calculations. The Ma = 50

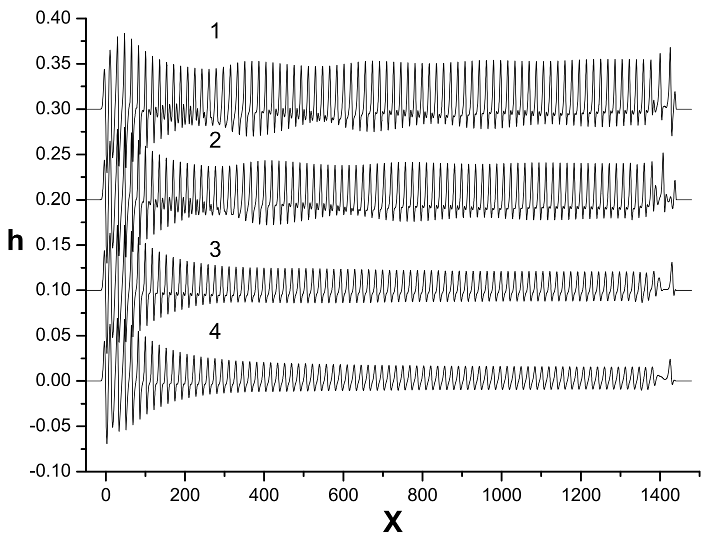

FIG. 13: Thick wall. β = 90°, ε = 0.1, Pr = 7, S = 1, Bi = 0.1, and Ma = 50. Here, ω = 0.5 and Re = 1.391 and the calculation time is T = 1000. d/Q_c is: (1) 0.1, (2) 1, (3) 10, and (4) 100.

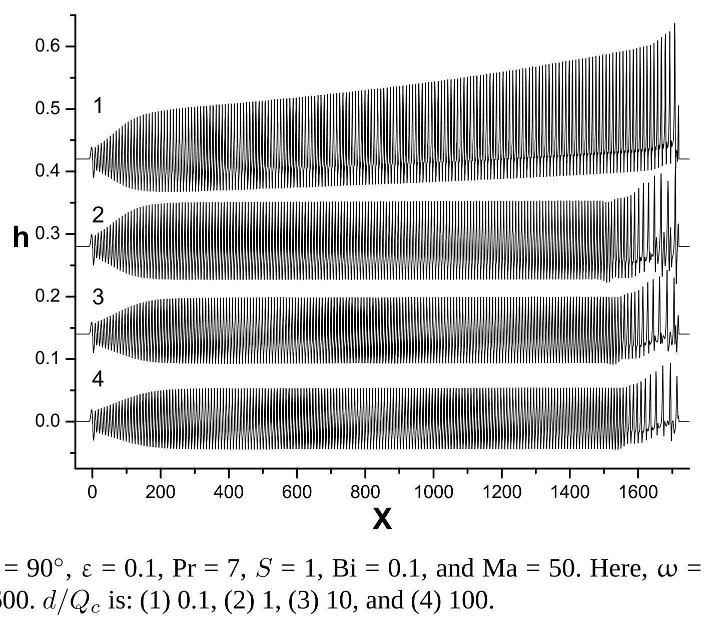

FIG. 14: Thick wall. β = 90°, ε = 0.1, Pr = 7, S = 1, Bi = 0.1, and Ma = 50. Here, ω = 2 and Re = 2.783 and the calculation time is T = 600. d/Q_c is: (1) 0.1, (2) 1, (3) 10, and (4) 100.

is relatively large and in Fig. 13, for ω = 0.5 with Re = 1.391, the free surface already shows wave modulation for small values of d/Q_c. The modulation disappears, increasing d/Q_c. At the same time, the amplitude decreases, as can be seen. Note that at d/Q_c = 100 the flow is very similar to the isothermal one. In Fig. 6 it is shown that for ω = 0.5 and Re = 1.391 the curves cross at a point slightly above subcriticality, for any magnitude investigated of d/Q_c. As can be seen in Fig. 13, saturation is very good.

An increase of the frequency to ω = 2 and the Reynolds number to Re = 2.783 leads to the results of Fig. 14. It is clear that for Ma = 50 and small d/Q_c the perturbations can not saturate even though the corresponding point in Fig. 6 is far above subcriticality. The figure shows that saturation can be attained for $d/Q_c \geq 1$. No modulation is observed and the amplitude decreases with d/Q_c. More numerical results can be found in Dávalos-Orozco (2012).

Among other conclusions (Dávalos-Orozco, 2012), two relevant ones are drawn from the numerical calculations. First, a very important result is that even for large magnitudes of the Marangoni number nonlinear saturation can always be attained for large enough magnitudes of d/Q_c where the flow is similar to the isothermal one. Second, the amplitude of the perturbations decreases with an increase of d/Q_c and the spatial modulation due to thermocapillarity disappears.

7. FLOW DOWN A CYLINDRICAL WALL

The problem of thin films flowing down cylindrical surfaces has been investigated by many researchers due to the large number of applications like the condensation of liquid in tubes, cooling of heat pipes, refrigeration, coating of curved surfaces, etc. Some of these are the subject of this review.

The effect of curvature in cylinders is reflected in the surface tension force term. This term depends on the inverse of the radius which represents the curvature of a section of the uniform cylinder. From the stability point of view, the growth of this term means increase of instability due to the throttling effect it has through surface tension, making the fluid free surface reduce its radius until film rupture. When an inviscid fluid column is in a hydrostatic state, a perturbation grows only when the product of the axial wavenumber times the radius has a magnitude smaller than 1. This is called the Rayleigh instability (or Rayleigh criterion). If the fluid shows viscous effects or is subjected to other body forces, this criterion is modified. However, it is important to take into account the Rayleigh instability due to its simplicity. The free surface curvature in the axial direction of the liquid column is stabilizing. When a main flow is present the criterion changes too. The main flow in the axial direction can be stabilizing for some magnitudes, helping to avoid the throttling effect of surface tension in the radial direction.

7.1 Flow Down Cylinders

One important motivation for this research is the coating of fibers or wires. The names of those cylindrical structures immediately suggest their thinness, that is, their small radius. This is of concern due to the action of the Rayleigh instability which is the cause of nonuniform coating.

The linear stability which occurs when coating a wire pulled out from a pool was investigated by Lin and Liu (1975b). Their results were used to compare with experiments. Krantz and Zollars (1976) made linear calculations and, from their results, pointed out that it is an error to make experiments in vertical cylinders to measure the instability corresponding to a flat plane. Homsy and Geyling (1977) calculated the stability of a film on a wire withdrawn from a pool. The main interest is to calculate the maximum coating thickness. They find that there is an optimal coating withdraw speed from the pool for stability. Shlang and Sivashinsky (1982) investigated for the first time the three-dimensional stability by means of an evolution equation calculated under the longwave approximation. They also assume a very strong surface tension and large radius of the cylinder. A condition is obtained for the appearance of azimuthal perturbations. However, it is shown that the axial mode is always the most unstable. The external perturbations like the effect of a countercurrent gas were taken into account by Bensalah and Brun (1986). They find that at low velocities the gas flow stabilizes.

The influence of condensation on the linear stability was investigated by Lin and Weng (1987). Condensation has a dual role but they found that the stabilizing effect dominates over the destabilizing one. Solorio and Sen (1987) made experiments to check the linear stability results. It is found that the flow is unstable for all Reynolds and Weber numbers and radius of the cylinder. The range of unstable wavenumbers and the maximum growth rate wavenumber increase with the decrease of the radius of the cylinder.

Rosenau and Oron (1989) calculated a nonlinear evolution equation with a restricted approximation of fast spatial changes. Their interest is to show that the numerical solutions of the free surface perturbations grow and break up at a finite time. Cheng and Chang (1992) investigated more deeply the instabilities found by Shlang and Sivashinsky (1982) and obtained a condition for the appearance of azimuthal modes when the surface tension is large. They show that these azimuthal waves are stable and only the axial perturbations persist. Besides, they investigate the side-band instability of the perturbations. Trifonov (1992) obtains a set of evolution equations using the integral boundary layer method. There are families of solutions which appear according to the variation of the parameters in the equations. The amplitude of surface perturbations increases when the radius decreases, with the possibility of rupture. Frenkel (1992) found an approximate evolution equation for flow on fibers by means of a multiparameter perturbation. This equation reduces to the Kuramoto-Sivashinsky equation when the maximum amplitude is small. The radius is small in this case and the amplitudes are large. It is found that the critical film thickness for rupture is proportional to the cube of the fiber radius. Frenkel (1993) proposes the calculation of an evolution equation using the lubrication approximation with a small wavenumber expansion. The result is a Benney-type equation in three dimensions and in cylindrical coordinates. When the radius tends to infinity the equation reduces to the Benney equation. Kerchman and Frenkel (1994) use periodic boundary conditions to investigate numerically the interaction of solitary wave pulses using the equation obtained in Frenkel (1992). It is found that pulses' collisions can be elastic and anelastic, with coalescence depending on the thickness of the layer. Tsvelodub (1994) obtains an evolution equation and investigates the spiral (not the azimuthal) perturbations instability in the case of small velocity. This equation is used by Tsvelodub (1995) to show that it is possible to find periodic steady solutions and that they are side-band unstable for small wavenumbers.

Hung et al. (1996) use a weakly nonlinear analysis to calculate an equation including condensation effects in two dimensions. They make a multiple scales analysis to understand the instability. It is found that the condensate film flow is more stable than isothermal flow. For any cylinder radius the velocity of the isothermal flow is larger than that of the condensate flow. Chen and Hwang (1996) investigate numerically an evolution equation in the longwave approximation in order to determine the conditions for rupture of the film. They find that reduction of the radius and the increase of surface tension accelerate that rupture in the presence of van der Waals forces. Experiments on turbulent thin film flow inside a cylinder have been investigated by Karami and Kawaji (1998). There is an important discrepancy between the measured velocity profile and the theoretical calculations. Good agreement is possible with smooth films. The wave-induced turbulence produces a flat velocity profile. They find that the large perturbation waves do not carry large fractions of the liquid mass. An extensive review of the flow down fibers is found in Quéré (1999).

The flow on fibers was investigated by Kornev and Neimark (1999). They include the adhesion between wall and fluid and the effect of the gas. The linear instabilities are caused by Kelvin-Helmholtz instability and by viscosity. They calculate the growth rate and estimate the rupture time. Chang and Demekhin (1999) investigate the formation of drops in a fiber. They use the model equation of Frenkel (1992) to find localized structures in the form of pulses which grow continuously. Use is made of matched asymptotic expansions to understand the development of the drop.

Experiments were performed by Dao and Balakotaiah (2000) on the problem of occlusion of the air space in flow down the inside of a cylinder. The radius of the cylinder is small and the results depend on the flow velocity and surface tension. Kliakhandler et al. (2001) present experimental results on the flow in a vertical fiber. They find three different flow regimes. They also develop a small Reynolds number model which agrees with the experiment. Kil et al. (2001) solve numerically a set of equations of the integral boundary layer type. The investigation is done only of flow in the inside of the cylinder with applications to air-cooled heat exchangers. The results agree for experiments with cylinders of diameters between 2 and 100 mm. The Reynolds numbers were between 20 and 100. Cheng et al. (2001a,b) present nonlinear analysis results of non-Newtonian thin films flowing down a cylinder. Gorla (2001) uses a generalized Newtonian fluid with power-law constitutive equation to investigate the rupture of the film. It is found that the rupture time for dilatant fluids is larger than for a Newtonian one and that it is smaller for a thinning fluid. Cheng et al. (2001a) investigate the flow of a viscoelastic Walters B fluid and Cheng et al. (2001b) investigate the stability of a micropolar fluid which has internal degrees of freedom. In both cases they obtain an evolution equation which is subjected to a multiple scales analysis to understand the stability.

The phenomenon of occlusion is investigated experimentally by Mousa et al. (2002). It is found that, contrary to the case of intermediate Reynolds number, for large Re the occlusion velocity increases with an increase of flow rate. Bocharov and Tsvelodub (2003) calculate a three-dimensional evolution equation in the longwave and large radius approximation. This equation reduces to the Kuramoto-Sivashinsky equation in the two-dimensional case. They found the possibility of solitary wave solutions in the numerical calculations.

Zuccher (2005) applies new experimental measurement techniques to investigate the wire coating instabilities. With this he is able to detect important wave characteristics. It is found that short waves dominate when the entrance speed is small. Longwaves dominate at high speeds. Jiang et al. (2005) make a correlation with experiments related to wire coating with the goal to understand the film behavior in optical fibers. They solve the linear stability problem and obtain a nonlinear equation which is solved numerically. When the wire drawing velocity exceeds a certain magnitude, the perturbation thickness may double that of the unperturbed film.

Sisoev et al. (2006) use the equation derived by Trifonov (1992) to compare its linear version with the linear Navier-Stokes equations results. The comparison was good. In the nonlinear numerical solution they found the existence of traveling waves. The formation of beads appears when the ratio of film thickness over fiber radius is large. Cheng and Liu (2006) present linear stability results of a power-law thin liquid film. It is found that the phase speed along with the unstable region increases and the growth rate decreases when the power-law index of the dilatant fluid increases. Calculations of flow under a magnetic field were done by Cheng and Liu (2007) and Cheng and Lin (2007), combined the influence of viscoelasticity and a magnetic field. The magnetic field is applied transverse to the film and works to increase the stable region and to decrease the growth rate. The viscoelastic case is nonlinear and uses a second-order fluid model. It is shown that viscoelastic effects destabilize but the magnetic field is still able to stabilize the flow. Experiments were performed by Moldavsky et al. (2007) exciting the film by means of ultrasound. The possibility of instability and flow rate control is detected. Even more, the possibility to stop the flow completely is found.

The aim of Jain and Shankar (2008) is to stabilize a thin film flowing down a cylinder by means of an elastic solid layer between the cylinder and the fluid. They investigate the linear stability and find that there are ranges of the parameters where the flow can be stable. However, this is not possible when the solid itself becomes unstable. Zuccher (2008) made experiments on the instability of the liquid on a wire pulled out from a pool and that is passed through an annular die. For small velocities it is found that only one perturbation wavenumber is present. However, for high velocities no wave appears. Smolka et al. (2008) made experiments using fluids of different densities and surface tensions. They found good agreement with theory for small wavenumber and Reynolds numbers but not very good for moderate Reynolds number. When the flow flux is smaller than a critical one, the perturbations appear at the same place of the wire. Above that critical value the perturbations are modulated and the position of appearance is irregular. Ruyer-Quil et al. (2008) use a small wavenumber and an integral boundary layer approximation (weighted-residuals approach) to understand the stability of the flow down fibers which are related with experimental results. Their approximations are drawn from the magnitudes of the parameters used in the experiments. A set of two evolution equations is obtained and solved numerically. The linear version of the equations was compared successfully with the Orr-Sommerfeld equation. The nonlinear solutions give traveling waves which also have branches of solutions. Moctezuma-Sánchez and Dávalos-Orozco (2008) investigate the linear three-dimensional instability of a viscoelastic fluid layer flowing down a cylinder. The model constitutive equations correspond to the Oldroyd B fluid model. A small wavenumber and large radius approximation is used. Under these conditions, the viscoelastic effects only appear in the form of the difference of the relaxation and retardation times. It is shown that these effects appear in two forms. First, the growth rate increases with viscoelasticity. Second, viscoelasticity promotes the appearance of more azimuthal modes than in the Newtonian case. However, the axial (two-dimensional) mode is always the most unstable one (as in the Newtonian fluid but with larger growth rate).

Zuccher (2009) shows experimental results of the stability of a liquid flowing down a wire pulled from a liquid bath subjected to an annular jet. The results are compared with an analytical model. It is found that the final thickness of the film increases with the capillary number and that it decreases with the nozzle stagnation pressure. When perturbations appear, the wavelength is of the order of the radius of the wire, but when they are detected the amplification factor is negative and the film is stable. Novbari and Oron (2009) use the energy integral method (Usha and Uma, 2004) to calculate a set of evolution equations for the film falling down a cylinder. They make calculations of the linear stability and find nonlinear traveling wave solutions of the nonlinear equations. There are two kinds of traveling waves depending on the magnitude of the parameters. Grünig et al. (2010) investigate the effect of a counterflow gas on the stability with the goal to understand, in an elementary example, the flow behavior of a wire bundle packing in the chemical industry. Up to certain counter gas flow the beads deform but for higher flows the beads disintegrate and flooding occurs.

Zhao et al. (2011) present the linear stability of a very thin film of 100 nm flowing down a fiber including van der Waals forces. It is shown that the van der Waals forces increase the wavenumber and the growth rate. They found that this forces promote the absolute instability and that the gravity force promotes the convective instability. Novbari and Oron (2011) use the energy integral method to find a set of evolution equations whose traveling wave solutions can bifurcate supercritically and subcritically. The subcritical case is sensitive to initial conditions but the supercritical one is not. Both present coexistence of a number of flows for the same set of parameters. Zakaria and Gamiel (2012) calculate the linear and nonlinear instability of a film flowing on a cylinder moving in the direction of its axis and in the presence of gravity. For the nonlinear problem use is made of multiple scales to obtain a Ginzburg-Landau equation. The Reynolds number is based on the cylinder velocity and plays a stabilizing role when it is large in the longwave approximation. At small values of the Weber number and nondimensional radius the flow is stable. Stability is also attained when We and Re are large. In the nonlinear case, the increase in the modulation wavenumber and the radius have stabilizing effects. Ruyer-Quil and Kalliadasis (2012) use the set of evolution equations obtained in Ruyer-Quil et al. (2008) for the problem of flow down a fiber. It is shown that the film is dominated by isolated waves or solitons even when the flow is disordered. They also found a number of regimes of the droplike traveling wave solutions.

7.2 Flow Down a Rotating Cylinder

When the liquid film flows down a vertical rotating cylinder, two new forces appear if the reference frame is located on the cylinder. They are the Coriolis force and centrifugal force. These forces change the instability with respect to the flow on a stationary cylinder. When the film is on the outside of the cylinder the centrifugal force destabilizes, making the crests of the free surface perturbations grow, and when it is in the inside of the cylinder the crests decrease and stabilize. The centrifugal force acts in the radial direction but the Coriolis force may have important effects in the azimuthal direction.

Iwasaki and Hasegawa (1981) made linear stability calculations when the radius and the Weber number are large. They found, besides the effects of the centrifugal force, that for small Re the neutral curve only depends on the radius and not on rotation. The phase velocity, which in their approximation only depends on the radius, decreases (increases) with the decrease (increase) of the radius. Linear stability calculations have been done by Dávalos-Orozco and Ruiz-Chavarría (1993) in the small wavenumber approximation and in the small Reynolds number approximation for any wavenumber. The main interest of this paper is to investigate the possibility of the appearance of azimuthal modes as the most unstable ones due to rotation. In the small wavenumber approximation the Coriolis force could not appear in the stability. For large radius asymptotic expressions were obtained. The growth rate depends linearly with respect to the nondimensional centrifugal force, Re and We and it decreases with the square of the radius. For flow in the inside of the cylinder a critical centrifugal number is calculated when this force balances the inertial and capillary forces. In the small Reynolds approximation the Coriolis force comes into play and also its effect on the azimuthal modes is present. It is shown that the flow in the outside of the cylinder can be unstable for the axial mode and the first azimuthal mode, but not for modes above the first one. For flow in the inside of the cylinder it is found that only the axial mode is the most unstable one. Ruiz-Chavarría and Dávalos-Orozco (1996) found an analytical condition to introduce the effect of the Coriolis force in the small wavenumber approximation and were able to understand its consequences in a wider range of the parameters. In Dávalos-Orozco and Ruiz-Chavarría (1993) is shown, by means of dimensional analysis, a relation between the nondimensional centrifugal force and the Taylor number representing the Coriolis force. Therefore, when both forces play a role in the instability it is only necessary to use the centrifugal number. The phase velocity in the absence of rotation is $c_0 = 2$. It is shown that the phase velocities of the first mode, in the inside and on the outside the cylinder, are smaller than this value and depend on the radius of the cylinder. They are decreasing when the radius decreases. The axial mode is stabilized by centrifugal force for flow in the inside and destabilized for flow on the outside. For relatively large radius, it is found that the effect of the Coriolis force is opposite to that of the centrifugal force with respect to the first azimuthal modes ± 1. It is stabilizing for flow on the outside and destabilizing for flow in the inside of the cylinder. For a radius smaller than a critical one, the Coriolis force effect adds to that of the centrifugal force on the outside and in the inside of the cylinder. It is shown for flow on the outside of the cylinder that the negative first azimuthal mode can be the most unstable one for some values of the parameters and in particular for large radius and small wavenumbers. For flow in the inside of the cylinder the most unstable mode is always the axial one. However, if it were possible to control the magnitude of the wavenumber, for small wavenumbers and some particular values of the other parameters, the first negative mode can be more unstable than the axial one. These findings were the motivation to investigate the possibility of the appearance of higher azimuthal modes as the most unstable ones. To attain this goal we need to widen the range of the parameters involved. Therefore, numerical analysis of the generalized system of the Orr-Sommerfeld equations was done by Ruiz-Chavarría and Dávalos-Orozco (1997). It is shown that a number of azimuthal modes may appear under rotation but only the axial and the positive and negative first azimuthal modes can be the most unstable in flow down the outside of the cylinder. Note that the condition to be the positive or the negative first mode depends on the relative direction of the rotation vector. In Fig. 15 the growth rates (in that paper it is a capital omega) of the two first modes are shown for different centrifugal parameters. From a comparison of the two Figures 15(a) and 15(b), it is clear that the first positive and negative modes are the most unstable at different rotation rates and at different regions of the wavenumber range investigated. Figure 16 shows contour plots of the growth rate against the wavenumber and the Reynolds number. If Figs. 16(a)–(c) were superposed, it is possible to find the unstable region where each mode is the most unstable one. It is clear that the corresponding regions are located at different areas of the (k, Re) plane. The unstable region of the axial mode is larger than that of the azimuthal ones. The maximum growth rate of the azimuthal modes is located at large Reynolds numbers and very small wavenumbers. In Fig. 7 of Ruiz-Chavarría and Dávalos-Orozco (1997) it is possible to see the increase of stable area (above the zero contour line) when the radius and the Weber number are reduced. These plots show the complex behavior of the film flow down a rotating cylinder. For flow in the inside of the cylinder only the axial mode is the most unstable one, but this does not prevent the appearance of azimuthal modes. However, all these modes are suppressed by an increase of the centrifugal force and the axial one prevails. Another example of the importance of the azimuthal modes is found in a related system of a stratified inviscid two-layer liquid inside a rotating annulus investigated by Dávalos-Orozco and Vázquez-Luis (2003).

Weakly nonlinear stability of a viscoelastic Walters B fluid layer flowing down a rotating cylinder was investigated by Chen et al. (2003). Only the axial mode was taken into account for flow in the outside of the cylinder. They fix the viscoelastic parameter to calculate the instability. Chen et al. (2004) investigate the nonlinear instability of a condensate film down the outside of a rotating cylinder. The nonlinear flow of a film on the inside of a rotating

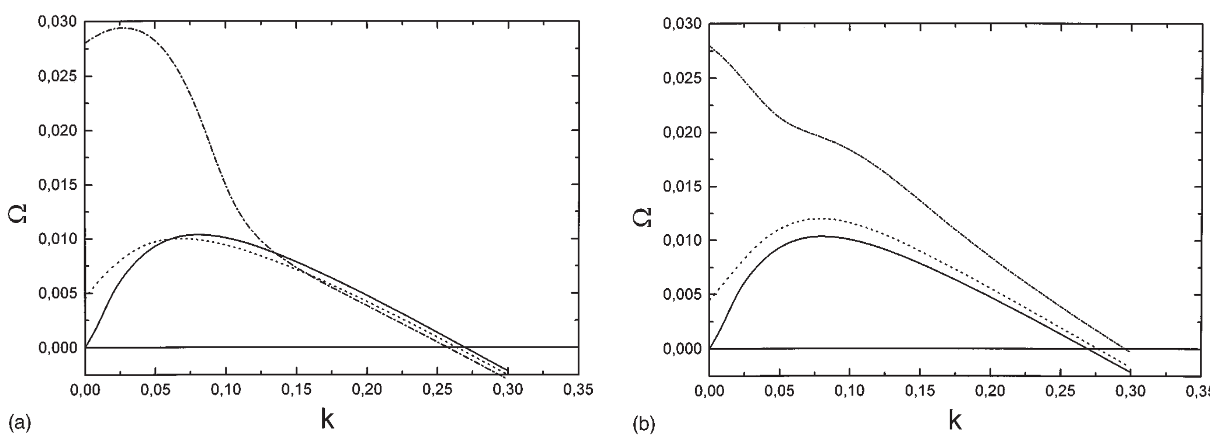

FIG. 15: Flow on the outside of the rotating cylinder. Growth rate against wavenumber. The adimensional radius is 7, We is 15, and for three centrifugal numbers: 0 (continuous line), 0.01 (dashed line), and 0.1 (dot-dashed lines). (**a**) Re = 100 and first positive azimuthal number, (**b**) Re = 100 and first negative azimuthal number.

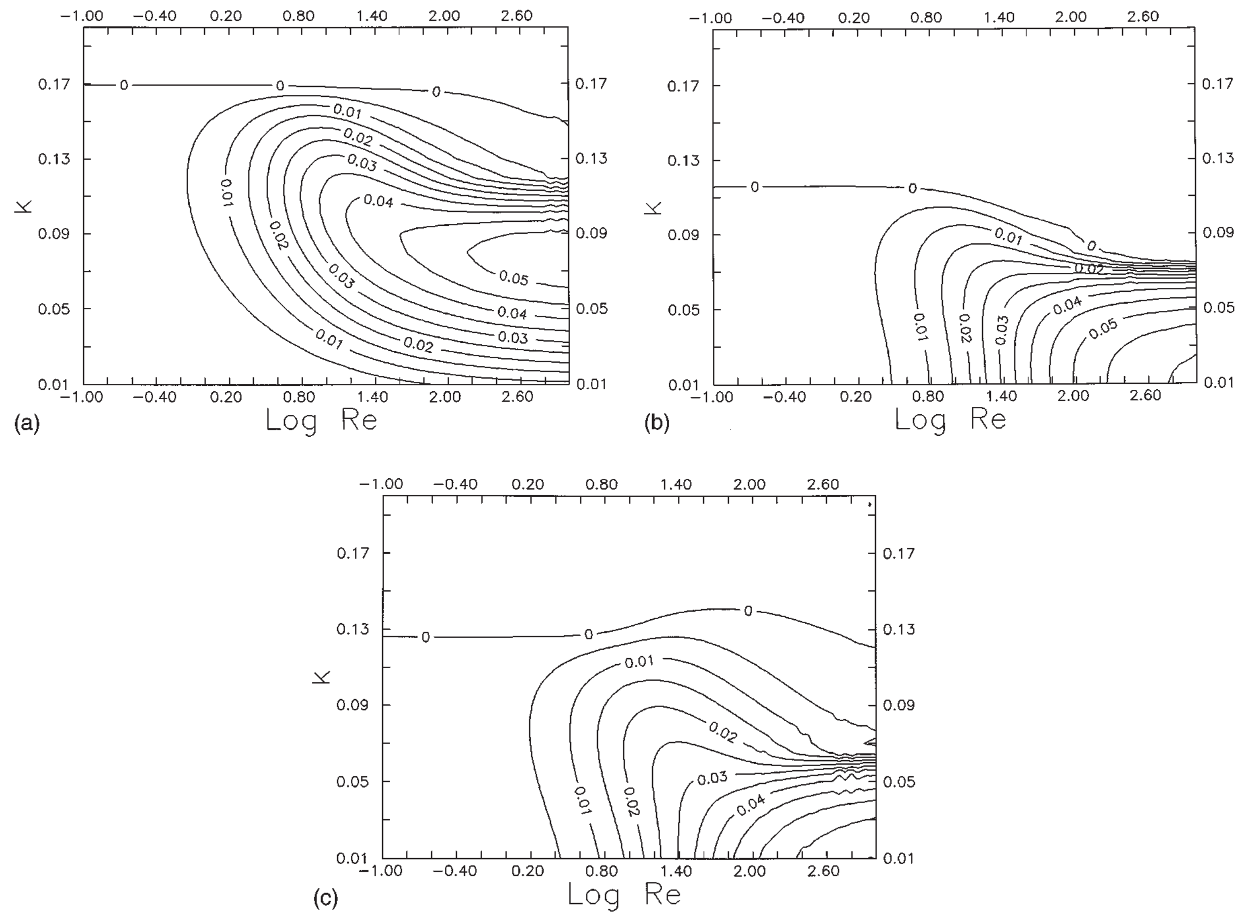

FIG. 16: Flow on the outside of the rotating cylinder. Contour plots of the growth rate against the wavenumber and Reynolds number. The We is 100, the adimensional radius is 10, and the centrifugal number is 0.5. The modes are (**a**) 0 (axial), (**b**) 1, and (**c**) –1.

cylinder was investigated by Chen et al. (2005a) and its non-Newtonian viscoelastic solution is given in Chen et al. (2005b). The nonlinear condensate film flow in the inside of a rotating cylinder is investigated by Chen et al. (2006). Cheng et al. (2011, 2012) combine the magnetic field with the rotation of the cylinder to investigate the nonlinear stability of the film. Cheng and Liu (2012) calculate the stability of a film of a non-Newtonian power-law fluid under the combined effects of magnetic field and rotation of the cylinder.

7.3 Flow Down a Heated Cylinder

The problem of heating has been used to present the effects of evaporation and condensation on the film flowing down a cylinder. The effects of thermocapillary forces have been neglected. Here, a review is given of the results calculated by Dávalos-Orozco and You (2000) on the linear stability of a film flowing down a vertical heated cylinder. The flows investigated are those in the inside and on the outside of the heated cylinder. As will be seen, in these two cases the stability is different for each of the axial and azimuthal modes (as was also shown in the flow down a rotating cylinder). The problem is divided into two parts: flow in the absence of gravity and flow on a vertical cylinder under gravity. The linear continuity, Navier-Stokes, and heat diffusion equations are assumed to have normal modes solutions. The resulting system of coupled equations is solved numerically to calculate the proper values of the problem.

7.4 Flow in the Absence of Gravity

In the absence of gravity the flow is due to the thermal Marangoni perturbation. The problem has the nondimensional parameters Ma, Bi, Pr, the radius, and the Crispation number Cr, which depends on the inverse of the surface tension. The maximum growth rate (the imaginary part of the frequency in that paper) for flow in the outside of the cylinder shows how the increase of the radius and the Marangoni number makes it possible to increase the number of azimuthal modes of instability. This number is considerably reduced by a decrease of the crispation number (increase of surface tension). From Fig. 17 it is seen that for a crispation number of magnitude 1 (continuous lines) it is possible to excite azimuthal modes until 13, after which a certain Marangoni number becomes the most unstable one [it crosses from below the curves of modes 0 (axial) and 1]. Notice that between the curves of modes 0 and 13, for convenience, the curves are not plotted for intermediate magnitude. Decreasing the crispation number to 0.001 only modes 0 and 1 are excited but mode 1 crosses from below the curve of mode 0 at some value of Ma. For the smaller crispation number only the mode 0 is the most unstable one in the range of the Ma investigated.

The flow in the inside of the cylinder presents a different panorama. In the same way as for flow on the outside the cylinder, more azimuthal modes are excited when the radius and the Marangoni number are increased. In the range of the parameters investigated, the appearance of mode 11 was found. However, in this flow only the zero mode (the axial one) can be the most unstable.

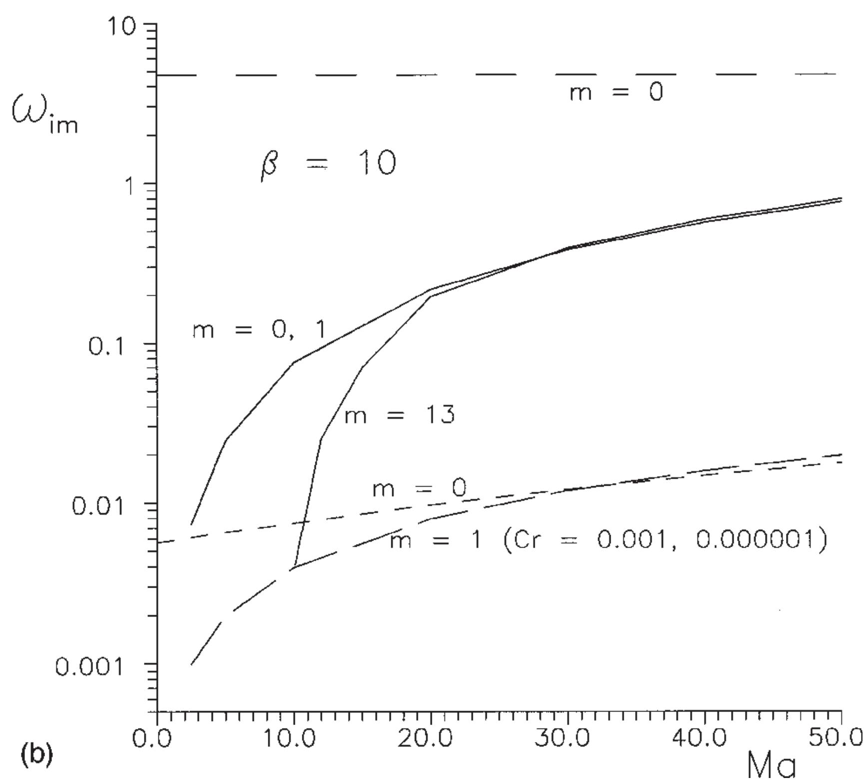

FIG. 17: No gravity. Flow on the outside of the heated cylinder. Maximum growth rate against Marangoni number. Pr = Bi = 10, the radius is 10, and the crispation number is 1 (continuous line), 0.001 (dashed line), and 0.000001 (large dashed line).

7.5 Flow Under Gravity

In this case, instead of the crispation numbers, the two new parameters are Re and We. Here, it is shown that from the point of view of the azimuthal modes, the Reynolds number has a stabilizing effect. In other words, for smaller Re it is possible to find higher azimuthal modes for a fixed Ma. An increase of Ma excites even higher azimuthal modes as shown in Fig. 18, which shows plots of the maximum growth rate against the Reynolds number. However, notice that the magnitude of the Marangoni numbers needed to excite modes 16 and higher are far smaller than in the absence of gravity for We = 10. In Fig. 18 the symbols dots, stars, and triangles show the Reynolds number for a change of mode. When Re is decreased the modes change from zero, one, two, etc, until a range of small Re appears where a large number of changes of mode exists (from 3 to 16) that are not possible to print for the sake of clarity. In the case Ma = 0, only the axial mode exists, as shown in the figure. If under the same conditions the radius is decreased to 1, fewer modes appear when Re is decreased. It is also shown that for Re small enough it is possible to stabilize the flow with a negative Marangoni number.

Calculations of the stability for flow in the inside of the cylinder show that for small enough Re very high azimuthal modes can be excited. However, only the axial mode is the most unstable one. If it were possible to control the perturbation wavenumber, there is some range of the parameters where some azimuthal modes can be the more unstable.

7.6 More References

Another problem where thermocapillarity changes the usual modes of (isothermal) instability is that of a moving liquid sheet subjected to a temperature gradient as investigated by Dávalos-Orozco (1999).

You (2002) investigated the possibility of feedback control of the instability of the film. He proposes to use an optical sensor to detect the displacement of the free surface and an actuator at the cylindrical wall. With this it is

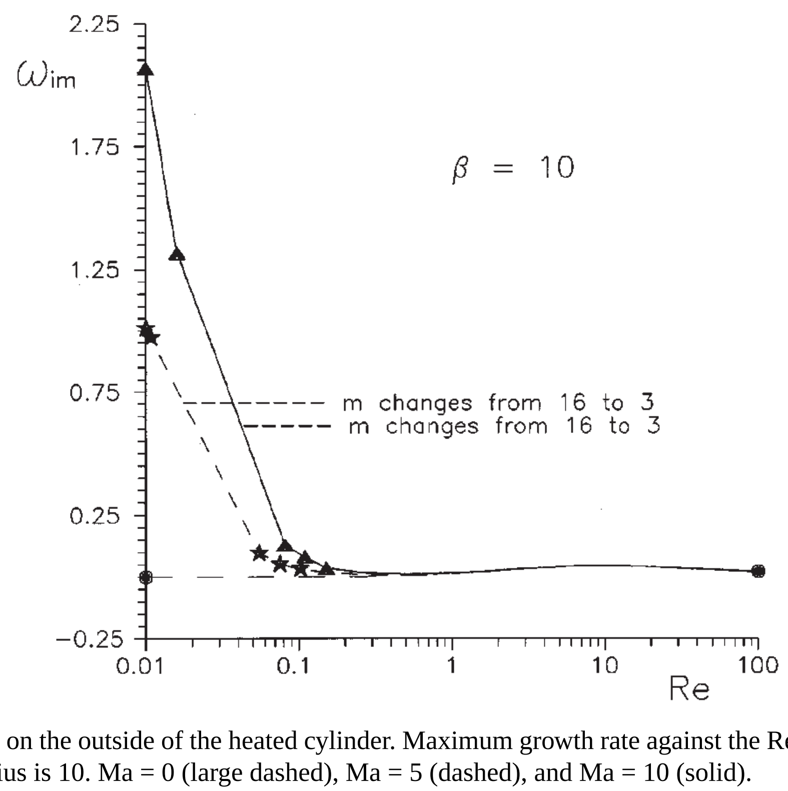

FIG. 18: Flow down on the outside of the heated cylinder. Maximum growth rate against the Reynolds number. We = Pr = Bi = 10, the radius is 10. Ma = 0 (large dashed), Ma = 5 (dashed), and Ma = 10 (solid).

possible to modify the wall temperature. It is shown, by means of numerical analysis of the linear system of equations in cylindrical coordinates describing the perturbations, that it is possible to suppress the perturbations' instability. He suggests that this method can also be used to destabilize the system on a particular purpose.

Wei (2005) made calculations of the thermocapillary linear and nonlinear instability of a core-annular flow. He assumes that the outer annular liquid layer is very thin and that it is heated or cooled according to the condition of a cooler core or a warmer core. It is found that Marangoni effects are stabilizing when the core is warmer than the annular liquid. The contrary occurs when the core is cooler and in this case the instability can be described by a Kuramoto-Sivashinsky equation if the surface tension is large. As found in Dávalos-Orozco and You (2000), the flow inside the heated cylinder has the axial mode as the most unstable one. Therefore, in the case of Wei (2005), for a very thin liquid annulus, the axial mode is also the most unstable one even though there is a core flow.

8. CONCLUSIONS

A survey of different problems on the flow of thin films flowing down walls has been presented. Some references are basic to understand phenomena described recently by new model equations.

A presentation of linear solutions has been given. This is relevant because, as explained in the discussion of the new model equations, one of the important tests of their validity is to compare with the solutions of the linear Navier-Stokes equations. The survey of the nonlinear models started with the Benney equation calculated under the lubrication approximation. This equation has been used in its different versions to include a variety of effects and boundary conditions. Comparison with numerical and experimental results shows the limitation of its solutions. This has motivated a number of researchers to look for improved approximations which may lead to new model equations simpler than the Navier-Stokes equations. The models described here were, among others, the Ooshida equation, the integral boundary layer set of equations, and the weighted-residual integral boundary layer set of equations.

The change of instability due to heating or cooling of the liquid film is reviewed in a separate section. In particular, the interest is focused on the nonuniform heating. This situation is more easily found in nature. However, it is shown that it also has important applications. The following section reviews the effect the thermal conductivity and thickness of the wall have on the stability of a falling film. Finally, the flow down a vertical cylinder is examined. This problem has interest in itself because of the wall curvature. It is well known that the theory of thin films flowing down flat walls is checked with experimental results of films flowing down cylinders. This is, for example, a motivation to investigate the flow down large radius cylinders. The effect of a heated cylinder on the thin film stability is reviewed too. It is shown that this new boundary condition radically changes the mode of instability of the cylindrical film.

The number of problems developed in this area is so huge that it is not possible to review all of them in these pages. Some of them are, for example, the effects of boiling in thin films, moving contact lines, wetting, and dewetting. Each one deserves a complete review and has to be seen as the element of an independent area of research, mainly because of their important industrial applications. It is the hope of this author that the present review will be of help as an introduction to the beginners and as a fast reference in their research to the experts.

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