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STABILITY OF THIN VISCOELASTIC FILMS FALLING DOWN WAVY WALLS

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In this paper the nonlinear stability of viscoelastic thin films falling down a smoothly deformed wall is investigated. The viscoelastic fluid satisfies the Oldroyd's nonlinear constitutive equation. Viscoelasticity adds new degrees of freedom to the fluid motion and it has been shown that the flow becomes more unstable than that of a Newtonian fluid. Here, it is shown that it is still possible to stabilize the viscoelastic thin film in a passive way by means of a spatial resonant effect due to the waviness of the wall, as done before for a Newtonian fluid in the small wave-number approximation.

KEY WORDS: *flow stability, thin films down walls, viscoelastic fluid, Oldroyd fluid, capillary instability, wavy wall*

1. INTRODUCTION

In industry it is common to coat surfaces with non-Newtonian liquids, which have different mechanical properties than Newtonian fluids. Some particular liquids are viscoelastic. These fluids are made of solutions of polymeric macromolecules which can deform due to the shear stress imposed on the fluid. This deformation makes the molecule take a different energy state which is not the minimum. When the stress is zero the macromolecules retract, taking with them a part of the surrounding fluid until they recover their minimum energy state. This is one reason why elasticity and viscosity are related. Besides, during the application of the shear stress the molecules may rotate and add spin to the surrounding fluid also. These new degrees of freedom add important properties to the viscoelastic fluid. It has been shown, under different flow conditions, that viscoelastic fluids are easier to destabilize and that their instabilities growth rates are larger than those of the Newtonian fluids [see, for example, Moctezuma-Sánchez and Dávalos-Orozco (2008)].

There are a variety of applications of viscoelastic fluid films. Some of them start from a hydrostatic state as in Wu and Chou (2005) and Espin et al. (2013). They investigate the use of viscoelastic films in lithographically induced self-assembly and their stability under electric fields. In particular Espin et al. (2013) assume both direct and alternate electric fields. In other cases, the films have an initial main flow. For example, the flow of long bubbles in tubes and the drag from a bath of a thin liquid film by a moving plate (Ro and Homsy 1995). Another instance is that of the shear stress effect on the acoustic response of two-layer films of viscoelastic fluids investigated by Voinova et al. (1999). Some practical uses of viscoelastic thin films are found in the pulmonary airway lining (Grotberg, 2001; Grotberg and Jensen, 2004). A viscoelastic model for the flow of that lining along with the spreading of surfactant is given by Zhang et al. (2002). The effect of the shear stresses and the behavior of a viscoelastic film with free surface before glass transition are investigated by Herminghaus et al. (2003). The problem of film rupture in liquid polystyrene films is investigated in Tsui et al. (2003) including van der Waal's forces. Dewetting and rupture is also the subject of the paper by Gabriele et al. (2006). These examples have the common feature of the instability of the films. The goal of this paper is to show that the topography assumed for the wall is able to stabilize in a passive way the main flow instability.

NOMENCLATURE

De	Deborah number
$\mathbf{D_R}$	deformation rate tensor
$\mathbf{e}$	shear rate tensor
f	total height of the free surface
g	gravity acceleration
$h(x,y,t)$	film local thickness
h_0	unperturbed film thickness
k	magnitude of the wave-number
k_c	critical wave-number
k_m	maximum growth wave-number
k_s	subcritical wave-number
L	ratio of wall and perturbation wave-numbers
L_1	relaxation time
L_2	retardation time
p	pressure
P_p	surface external pressure
Re	Reynolds number
S	scaled surface tension number
T	numerical calculation time
u	velocity x-component
v	velocity y-component
w	velocity z-component

Greek Symbols

β	wall inclination angle
ε	wave slope smallness parameter
ζ	wall deformation
κ	thermal diffusivity
λ	wavelength
ν	kinematic viscosity
ρ	fluid density
σ	surface tension
Σ	surface tension number
τ	shear stress tensor
τ_{ij}	shear stress tensor components
ω	frequency of oscillation

Here, in particular, the instability of thin viscoelastic films flowing down wavy walls is investigated under the small wave-number approximation. Several papers which form part of the background in this area of research are discussed in the present section.

First, the review begins with references related with Newtonian flows. The assumption of a series solution of the equations of motion within the small wave number approximation, led Benjamin (1957) to calculate the linear stability results. His method is very cumbersome and was modified and simplified by Yih (1963). He includes the surface tension in the small wave-number approximation assuming it is large, as in many practical fluids, and also presents the results of the small Reynolds number approximation. The nonlinear equation, the Benney equation, was calculated by Benney (1966) under the small wave-number approximation. Gjevik (1970) introduced a normal modes series solution for the amplitude of the Benney equation to understand the nonlinear instability of the film. He obtained the formula which describes the nonlinear curve of subcriticality, below which the waves can not saturate. This formula has to be added to those of criticality and maximum growth rate calculated by Yih (1963) to understand the region of validity of the Benney equation. Pumir et al. (1983), by means of the theory of dynamical systems, showed for the first time the possibility of solitary wave solutions of the Benney equation. Numerical analysis using spectral methods was done by Joo et al. (1991a) in two dimensions and by Joo and Davis (1992) in three dimensions. It is shown in Joo et al. (1991b) that the Benney equation solutions present film breaking when the surface tension is small. Dávalos-Orozco et al. (1997) made numerical analysis in space and time to understand the evolution of the solutions of the Benney equation in the wave-number and Reynolds number plane. A Benney-type equation was obtained by Dávalos-Orozco and Busse (2002) to describe the evolution of the perturbations of a thin film falling down a rotating wall.

A Benney-type equation was calculated to investigate the flow instability of a thin film falling down an inclined wavy wall by Dávalos-Orozco (2007). Numerical calculations were made in space and time to better understand the behavior of the free surface perturbations. It is found that the instability can be suppressed when spatial resonance exists at a particular relation between the perturbation wave-number and wall wave-number. Under this resonance the film response to the wall waviness increases its amplitude in such a way that in some regions the film is very thin. In the thinner parts of the film, the local Reynolds number, which depends on the thickness of the layer, is very small and the perturbation stabilizes locally and temporally. In the higher part of the response the perturbation may destabilize strongly. However, it cannot do so because the perturbation cannot make it from the valley to the crest due to the strong stability effect in the thinner part. Therefore, the perturbation fades away in space and time. In Dávalos-Orozco (2007) the wall deformations have an infinite extension; however, it is of interest to investigate the effect of finite wall deformation on the free surface perturbations as done by Dávalos-Orozco (2008). In that paper, it is shown that even with long enough but finite wall waviness it is possible to stabilize the perturbations. If the wall has a small hole or pit it is possible to reduce considerably the perturbation amplitude for a large interval in space and time. Trifonov (1998, 2004, 2007a,b) also has investigated this problem using a different approximate system of equations. In his last two papers, he also found the stabilizing resonant effect of the wavy wall. The problem of wall waviness has also been investigated by other authors like Bontozoglou and Papapolymerou (1997), Malamataris and Bontozoglou (1999), Vlachogiannis and Bontozoglou (2002), Wierschem et al. (2002), Wierschem and Aksel (2003), and Scholle et al. (2004).

Heining et al. (2009) used the set of equations derived from the boundary layer integral method. They find bistable resonance when the slope of the wavy wall is large. Heining and Aksel (2010) use the same method to obtain a set of equations for a power-law fluid. They find that a shear thickening fluid is easier to stabilize by space resonance. Wierschem et al. (2010) investigate the stability of the vortices formed inside the valleys of the wavy wall. It is found that with the increase of the Reynolds number vortices appear, but for a further increase they disappear, reappearing with an extra increase. Nguyen and Bontozoglou (2011) find that, in their flow configuration, a large slope of the wall waviness can lead to flow separation. It is found that for this waviness two solution branches can coexist.

The so-called inverse problem is investigated by Heining and Aksel (2009) who reconstruct the unknown bottom topography based on the observed free surface deformation and by Heining (2011) who reconstructs the velocity field when the wall deformations are unknown. Different effects have been introduced such as porosity in the wall by Pascal and D'Alessio (2010) and the effect of an electric field by Pak and Hu (2011) and Veremieiev et al. (2012). Improved systems of equations have been calculated by Oron and Heining (2008) and by Hacker and Uecker (2009). Besides, Heining et al. (2012) investigate the mixing effect of the wavy wall.

Viscoelasticity has been taken into account in thin films flowing down walls since many years ago. The linear stability of one of the most simple fluids, the second-order viscoelastic fluid, has been investigated by Gupta (1967). The linear stability of a fluid satisfying the Oldroyd's model equations was investigated by Gupta and Rai (1967). Almost at the same time, the same constitutive equations were used by Lai (1967). Shaqfeh et al. (1989) show the possibility of a purely elastic instability in the falling film when the Reynolds number is very small. The nonlinear instability of an Oldroyd fluid with a power-law fluid term was investigated by Joo (1994) in the small wave-number approximation. Kang and Chen (1995) calculated a nonlinear evolution equation for an Oldroyd fluid but an order higher than that of Joo (1994). Uma and Usha (2004) take into account the effect of evaporation on the stability of a Walters B fluid. The linear stability of the same fluid was investigated by Sadiq and Usha (2005) when the film is heated nonuniformly from below. The stability of a Maxwell fluid layer with the effects of surfactant and shear on the free surface was calculated by Wei (2005). A study of the nonlinear instability of a Walters B fluid is presented in Uma and Usha (2006). Khayat and Kim (2006) made a numerical analysis of the flow of an Oldroyd viscoelastic thin film flowing on the surface of a cylinder. Moctezuma-Sanchez and Dávalos-Orozco (2008) calculated the linear three-dimensional small wave-number stability of an Oldroyd fluid falling down a cylinder. They show that increasing the viscoelastic parameter, the unstable region of the azimuthal modes increases considerably and that the growth rate of the perturbations increases notably, too. Even though more azimuthal modes destabilize in the viscoelastic problem, it is found that the most unstable mode is the axial one. The nonlinear calculations for a Walters fluid flowing down a nonuniformly heated wall were investigated by Mukhopadhyay and Haldar (2010). Amatousse et al. (2012) use the weighted residual integral method to obtain a set of evolution equations to describe the propagation of nonlinear traveling waves on the surface of a film of Walters B fluid.

In this paper, the instability of an Oldroyd fluid is investigated in the small wave-number approximation. It will be shown that, even though viscoelastic thin liquid films are more unstable than the Newtonian fluids, it is still possible to stabilize them in a passive way when flowing down a sinusoidal wavy wall.

The paper is organized as follows. In Sec. 2, the basic equations are presented in nondimensional and scaled form. Then the resulting evolution equation is given including the terms corresponding to a wavy wall. Section 3 contains the numerical results of the nonlinear stability and Sec. 4 presents the conclusions.

2. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

The system under investigation is a thin viscoelastic fluid layer flowing down an inclined wall with wavy deformations. The system is sketched in Fig. 1 where the flow is in the positive x-direction. In this paper the wall is set parallel to gravity and therefore the figure has to be rotated 90°.

The basic equations are formed by the the Navier-Stokes equations, the continuity equation, and the constitutive equations of the Oldroyd fluid. These equations are scaled selecting a smallness parameter which is the ratio of the wave amplitude over the representative wavelength. As can be seen this number is also representative of the slope of the wave, that is, $\varepsilon = 2\pi h_0/\lambda \ll 1$, where the thickness of the layer is h_0 and λ is the wavelength. For the space variables, h_0 is used in the z-direction perpendicular to the wall and $\lambda/2\pi$ in the x and y-directions (see Fig. 1). The time is made nondimensional with $h_0\lambda/(2\pi\nu)$. A combination of these quantities is used for pressure and velocity, that is, $\rho\nu^2/h_0^2$ and ν/h_0, respectively. Here, ν and ρ are the kinematic viscosity and density of the fluid, respectively.

In order to include the effects of the wavy wall in the boundary conditions, in nondimensional form they are evaluated at $z = \zeta(x, y)$, at the solid wall, and at $z = \zeta(x, y) + h(x, y, t)$. The function $\zeta(x, y)$ is the smooth profile of the deformed wall, which in the numerical analysis of Sec. 3 will be assumed sinusoidal.

The pressure is p and the three components of the velocity are (u, v, w) in the (x, y, z) directions, respectively. The angle of inclination of the wall is β. The Reynolds number is defined as $\mathrm{Re} = gh_0^3/\nu^2$. Thus, the equations of balance of momentum and continuity in nondimensional and scaled form are

$$\varepsilon u_t + \varepsilon u u_x + \varepsilon v u_y + w u_z = -\varepsilon p_x + \varepsilon \left(\tau_{xx}\right)_x + \varepsilon \left(\tau_{xy}\right)_y + \left(\tau_{xz}\right)_z + \mathrm{Re}\sin\beta, \tag{1}$$

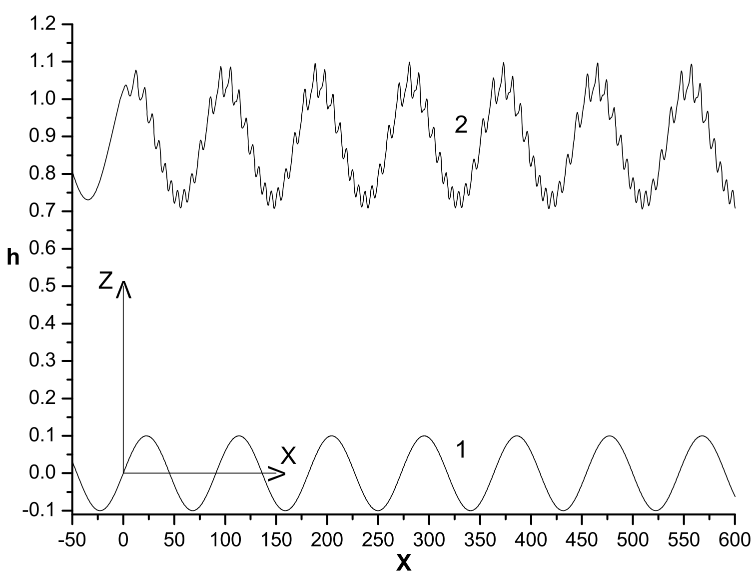

FIG. 1: Sketch of the physical system. The mean value of the main flow is in the x-direction, the same as the mean height of the wall. The mean height of the film is at $z = 1$ if it were flowing down a flat wall. In fact, the unperturbed film has a response to the wall deformation. In the vertical axis, h shows the total thickness of the fluid layer. The wall is parallel to gravity and therefore the figure has to be rotated 90°. (1) Wall sinusoidal deformation. (2) Time-dependent perturbations propagating on the free surface response to the wall deformation. The time-dependent perturbations are applied at $(x, z) = (0, 1)$. Notice the response of the film alone to the left of this point.

$$\varepsilon v_t + \varepsilon u v_x + \varepsilon v v_y + w v_z = -\varepsilon p_y + \varepsilon \left(\tau_{yx}\right)_x + \varepsilon \left(\tau_{yy}\right)_y + \left(\tau_{yz}\right)_z , \tag{2}$$

$$\varepsilon w_t + \varepsilon u w_x + \varepsilon v w_y + w w_z = -p_z + \varepsilon \left(\tau_{zx}\right)_x + \varepsilon \left(\tau_{zy}\right)_y + \left(\tau_{zz}\right)_z - \mathrm{Re}\cos\beta, \tag{3}$$

$$w_z = -\varepsilon u_x - \varepsilon v_y, \tag{4}$$

where the subindices x, y, z, and t mean partial derivatives. The boundary conditions at the wall and at the free surface are the no-slip condition:

$$u = v = w = 0, \quad \text{at} \quad z = \zeta(x, y), \tag{5}$$

the normal stress boundary condition:

$$-p + \frac{1}{N^2}\left[\varepsilon^2 \tau_{xx} f_x^2 + 2\varepsilon^2 \tau_{xy} f_x f_y - 2\varepsilon \tau_{xz} f_x + \varepsilon^2 \tau_{yy} f_y^2 - 2\varepsilon \tau_{yz} f_y + \tau_{zz}\right] = P_p(x, y, t),$$
$$-\frac{3}{N^3} S[(1+\varepsilon^2 f_y^2) f_{xx} + (1+\varepsilon^2 f_x^2) f_{yy} - 2\varepsilon^2 f_x f_y f_{xy}], \quad \text{at} \quad z = \zeta(x, y) + h(x, y, t), \tag{6}$$

where $N = \sqrt{1 + \varepsilon^2 h_x^2 + \varepsilon^2 h_y^2}$ and $f(x, y, t) = \zeta(x, y) + h(x, y, t)$. The first tangential stress boundary condition,

$$\varepsilon\left(\tau_{xx} - \tau_{zz}\right) f_x^2 + 2\tau_{xy} f_x f_y + \tau_{xz} f_x \left(\varepsilon^2\left(f_x^2 + f_y^2\right) - 1\right) + \varepsilon\left(\tau_{yy} - \tau_{zz}\right) f_y^2$$
$$+ \tau_{yz} h_y \left(\varepsilon^2\left(f_x^2 + f_y^2\right) - 1\right) = 0, \quad \text{at} \quad z = \zeta(x, y) + h(x, y, t), \tag{7}$$

and the second tangential stress boundary condition

$$\varepsilon\left(\tau_{yy} - \tau_{xx}\right) f_x f_y + \varepsilon \tau_{xy}\left(f_x^2 - f_y^2\right) + \tau_{xz} h_y - \tau_{yz} f_x = 0, \quad \text{at} \quad z = \zeta(x, y) + h(x, y, t). \tag{8}$$

The τ_{ij}, where the subindices i and j have the values x, y, and z, are the components of the symmetric shear stress tensor τ which satisfies the following Oldroyd constitutive equation:

$$\tau + L_1 \frac{D\tau}{Dt} = \mathbf{e} + L_2 \frac{D\mathbf{e}}{Dt}, \tag{9}$$

The time derivative is defined as the following upper convected time derivative:

$$\frac{D\tau}{Dt} = \frac{\partial \tau}{\partial t} + \mathbf{u} \cdot \nabla\tau - \mathbf{D_R} \cdot \tau - \tau \cdot \mathbf{D_R^T}, \tag{10}$$

where L_1 and L_2 are the adimensional relaxation and retardation times, respectively. The $\mathbf{e}$ is the shear rate tensor. The $\mathbf{D_R}$ is the deformation rate tensor and $\mathbf{D_R}^T$ is its transpose. The function $P_p(x, y, t)$ is a time dependent free surface pressure perturbation. This Eq. (9) in fact represents six coupled nonlinear differential equations due to the symmetry of τ. It is used in a scaled form to calculate the nonlinear evolution equation, as done by Joo (1994), but including the new boundary condition at the wavy wall.

The kinematic boundary condition is

$$w = \varepsilon h_t + \varepsilon u f_x + \varepsilon v f_y, \quad \text{at} \quad z = \zeta(x, y) + h(x, y, t). \tag{11}$$

An important assumption is that the liquid has a strong surface tension σ. Thus, $\Sigma = \sigma h_0/(3\rho\nu^2)$, the surface tension number, is changed into the scaled $S = \varepsilon^2\Sigma$, which is of order one.

In Eq. (5) the no-slip boundary condition is assumed in the deformed wall, as in the Newtonian fluid [see Dávalos-Orozco (2007)]. The interest on slip boundary conditions in non-Newtonian fluids arose many decades ago. Experimentally it may appear when the fluid presents dilatant power-law behavior. When the shear rate is strong its behavior becomes that of a solid slipping on the wall. The viscoelastic fluid slip on the wall may appear when the polymer molecules are entangled [see, for example, Gay (1999)]. However, in this case other constitutive equations have to be used to describe the flow. The Oldroyd's fluid model used in this paper lacks the terms needed to describe flow slip on the wall. To allow for slip the Oldroyd's model requires two more constants, one multiplying a second-order time derivative of the shear stress tensor and another one multiplying the second time derivative of the shear rate tensor [see Tanner and Walters (1998), p. 31]. Note that two models, one for weak slip and another one for strong slip, are proposed by Blossey et al. (2006) for a thin film of an Oldroyd fluid. In this sense, the no-slip boundary conditions presented in Eq. (5) are in agreement with the constitutive Eq. (9).

In the small wave-number approximation it is usual to expand the variables in terms of the small parameter ε. Assuming that the z-component of the velocity is very slow under the lubrication approximation, the expansions needed are

$$u = u_0 + \varepsilon u_1 + \cdots, \quad v = v_0 + \varepsilon v_1 + \cdots,$$
$$w = \varepsilon(w_1 + \varepsilon w_2 + \cdots), \quad p = p_0 + \varepsilon p_1 + \cdots. \tag{12}$$

All the components of velocity, the pressure, and the shear stress tensor depend on (x, y, z, t). The free surface deformation h and time-dependent free surface pressure perturbation P_p depend on (x, y, t). The problem will be restricted to two-dimensional flow. The reason is that, as will be shown presently, the wall waviness is expected to stabilize the flow in a short distance from the origin in such a way that the flow instability cannot grow into three dimensions. Then, the expanded variables, Eqs. (12), are introduced into the equations of motion and boundary conditions. The resulting equations are then solved order by order. At first order, the substitution into the kinematic boundary condition leads to the evolution equation of the free surface perturbations. That is,

$$h_t + \mathrm{Re}\sin\beta h^2 h_x + \varepsilon\left\{(\mathrm{Re}\sin\beta)^2\left(\frac{2}{15}h^6 h_x\right)_x + \left(\mathrm{Re}^2\mathrm{De}h^4 h_x\right)_x\right.$$
$$\left.+\frac{1}{3}\left[h^3\left(-\mathrm{Re}\cos\beta(\zeta + h)_x + 3S(\zeta + h)_{xxx} - (P_p)_x\right)\right]_x\right\} = 0. \tag{13}$$

Here, $\mathrm{De} = (L_1 - L_2)/3$ is the Deborah number, that is, one third of the positive difference between the relaxation and retardation times. As can be seen in Eq. 13, the viscoelastic term does not include ζ, which only appears in the terms found in the Newtonian problem by Dávalos-Orozco (2007). In this sense, if ζ is zero the equation reduces to that of Joo (1994) (but without the power-law fluid effects). If De is zero the equation reduces to that found by Dávalos-Orozco (2007). If both De and ζ are zero, Eq. 13 reduces to that of Benney (1966) [see also Joo et al. (1991a) and Dávalos-Orozco et al. (1997)].

The linear stability was investigated by Joo (1994) in the case of a vertical wall which corresponds to $\beta = 90°$. It is shown that the phase velocity is equal to Re $\sin\beta$. He also calculated the critical k_c, maximum growth rate k_m, and nonlinear subcritical k_s wave-numbers. It is shown that in the viscoelastic problem they satisfy the same relation as in the Newtonian fluid. That is,

$$k_c^2 = \frac{\mathrm{Re}^2}{S}\left(\frac{2}{15} + \mathrm{De}\right) = 2k_m^2 = 4k_s^2. \tag{14}$$

Some examples of plots of these curves are presented in Fig. 2 for $\beta = 90°$ and different values of De and ω, the frequency of the perturbation. They are shown in the (k, Re) plane. Notice that, because the Reynolds number is the phase velocity, the curves of the hyperbolas $k = \omega/\mathrm{Re}$ are also plotted in the figure for four different frequencies of oscillation. In the plane, they are the reference to find the wave-number corresponding to a given Reynolds number. The curves of criticality separate the regions of stable and unstable flows. The flow is stable above the straight continuous lines. The wave-number of subcriticality k_s is calculated using a normal mode expansion of the surface perturbation $h(x, t)$ in Eq. (13). The nonlinear ordinary differential equations for the amplitudes are solved for each order of the expansion and at second order k_s is the critical condition for which the nonlinear perturbation growth is zero. Under this approximation, it is supposed that below k_s the perturbations cannot have nonlinear saturation (Gjevik, 1970). Therefore, nonlinear saturation is attained only between k_c and k_s. However, it is shown numerically that the perturbations of the full Eq. (13) saturate below but not far from the curve of subcriticality k_s (Dávalos-Orozco et al., 1997). Despite this result, the curve of subcriticality is an important reference for nonlinear saturation. Thus, Fig. 2 will be used in the numerical calculations of the nonlinear problem. Note that the thick dotted line corresponds to the maximum growth rate of the Newtonian fluid, which is used as reference in the nonlinear results of Sec. 3.

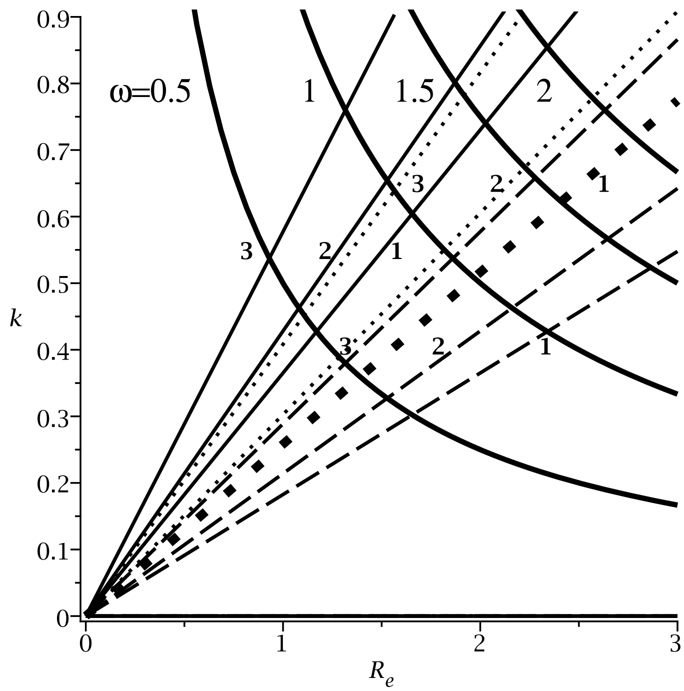

FIG. 2: $\beta = 90°$. Plots of k vs. Re for different De and ω; k_c continuous lines, k_m dotted lines and k_s dashed lines. For each De the number corresponds to (1) De = 0 (Newtonian), (2) De = 0.05, (3) De = 0.2. The thick dotted line corresponds to the maximum growth rate of the Newtonian fluid, which is used as reference. The thick continuous lines correspond to the hyperbolas $k = \omega/\text{Re}$ for different frequencies $\omega = 0.5$, 1, 1.5, 2.

3. RESULTS OF NUMERICAL ANALYSIS

For the numerical calculations in space and time of Eq. (13) an external time-dependent perturbation is added to control the evolution of the free surface waves. The external perturbation is represented by

$$P_p(x, y, t) = A \left| \sin \frac{\omega}{2} t \right| \exp[-a(x^2 + y^2)], \tag{15}$$

which appears in the normal stress boundary condition Eq. (6). It is assumed that it is an external pressure due to a turbulent air jet which strikes periodically at the origin on the free surface [see Lacanette et al. (2006)]. The constants which appear in Eq. (15) will be taken as $A = 0.0001$ and $a = 0.05$. The selection was made due to the sensitivity of the thin film instability to the parameters. For larger magnitudes, the initial perturbation amplitude at $x = 0$ increases in space and time without saturation, or else, the initial perturbation is so large that despite the decrease of the wave amplitude, it cannot reach its saturation amplitude in the space and time ranges of the numerical calculations.

The frequency of oscillation ω is divided by 2 because a jet has no suction and it is effective only when it strikes again on the surface. The wall is assumed sinusoidal and of the form $\zeta(x) = 0.1 \sin[x\omega/L\text{Re} \sin\beta]$. As can be seen in Eq. (13), the phase velocity of the perturbation is Re sin $\beta = \omega/k$, which is used to find a relation between the wall and the perturbation wavelengths. That is, if $\omega/L\text{Re} \sin\beta$ is the wave number of the wall, then L is the ratio between the wavelengths of the wall and the perturbation. This L is used in all the results to recognize the magnitude of that relation when spatial resonance is effective.

The numerical results will be presented starting with the solutions of flow down a flat wall and then the results of the wavy wall for two different values of L. The last of them will show the stabilizing effects of spatial resonance. For the given frequencies, the Reynolds numbers used will correspond to those of the maximum growth rate of the

Newtonian fluid. The points in the (k, Re) plane correspond to the intersections of the ω parabolas with the large dotted line of Fig. 2.

The first results are given for $\omega = 0.5$ and Re $= 1.391$ and are shown in Fig. 3 for flow down a flat wall. For the sake of clarity, only a section of the full numerical runs is presented in space from –50 to 600 units. It is shown that the film destabilizes considerably increasing the amplitude when De increases. The perturbation growth is very fast for large De. Note from Fig. 2 that for De $= 0.2$ this flow is below the curve of subcriticality, but saturation is attained.

Figure 4 presents results for the wavy wall and the same values of De. The left figure is for $L = 6$ and the right one is for $L = 4$. It is clear for $L = 6$ that the free surface perturbations propagate on the free surface response to the wall deformations without fading away in time and space [see Dávalos-Orozco (2007) for examples of free surface response to the wall deformations]. Their amplitudes are large as can be seen in the scale of the vertical axis of the graph. However, when $L = 4$ the perturbations fade away in a short interval of space and time, even for De $= 0.2$ which is below subcriticality. This means that spatial resonance starts to appear for a magnitude of L between 6 and 4. However, for this frequency and $L = 4$, Fig. 4 shows that resonance is very effective because, for all cases, the perturbations cannot survive two wave responses of the film.

Figure 5 shows results for the flat wall with a larger frequency $\omega = 1$ and Re $= 1.967$. The perturbation amplitudes are larger as can be seen looking at the scale. This has an important influence in the flow on a wavy wall as shown on the left Fig. 6 for $L = 7$. The perturbations decrease a little for small De but for De $= 0.2$ they remain almost the same running on the free surface response. However, for $L = 4$ the right figure presents the complete disappearance of the perturbations. They cannot even survive a half wave of the film response.

Figure 7 presents the numerical results of the flow down a flat wall for $\omega = 1.5$ and Re $= 2.41$. The amplitudes increase even more when De increases. The reason is that in Eq. (13) the viscoelastic term is also multiplied by the square of Re. Therefore, the Reynolds number supports the viscoelastic instability. In contrast, from the point of view of L, the left side of Fig. 8 shows that these perturbations are easier to stabilize for the Newtonian and De $= 0.01$ cases when $L = 9$. This is due to the double role played by the Reynolds number in the corresponding region of the (k, Re) plane. As seen in Fig. 2 the frequency and Reynolds number of these calculations cross at a point which is far above the curves of subcriticality, except for De = 0.2. As a consequence, the flows of the other De are easier to

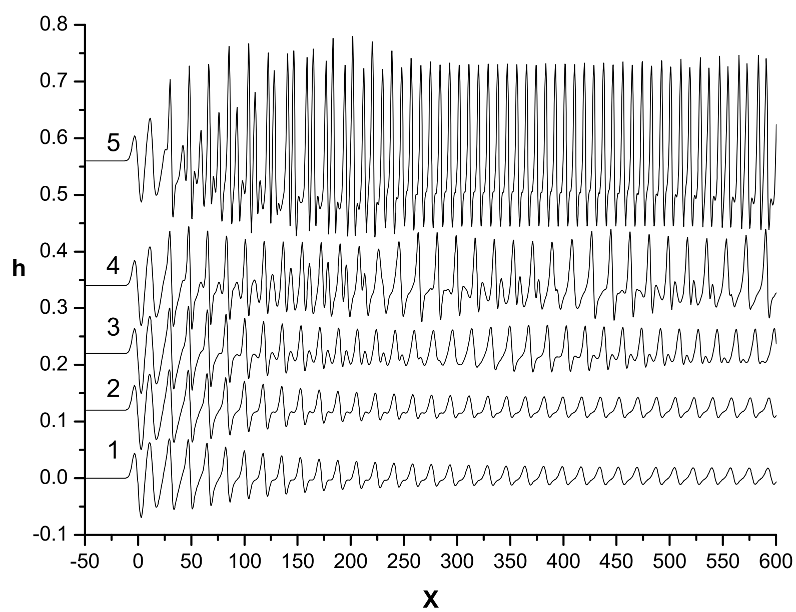

FIG. 3: Flat wall, $\beta = 90°$; surface perturbations for $\omega = 0.5$ and Re $= 1.391$ and $T = 1000$: (1) De $= 0$ (Newtonian), (2) De $= 0.01$, (3) De $= 0.05$, (4) De $= 0.1$, (5) De $= 0.2$.

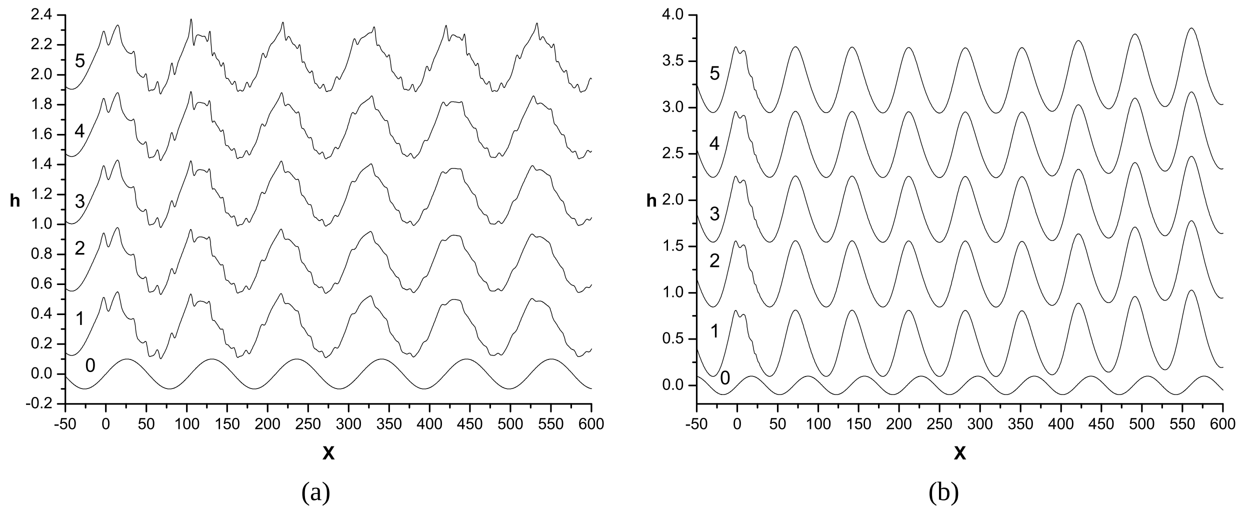

FIG. 4: Wavy wall, left $L = 6$ and right $L = 4$, $\beta = 90°$; surface perturbations for $\omega = 0.5$, Re $= 1.391$ and $T = 1000$: (0) wall, (1) De $= 0$ (Newtonian), (2) De $= 0.01$, (3) De $= 0.05$, (4) De $= 0.1$, (5) De $= 0.2$.

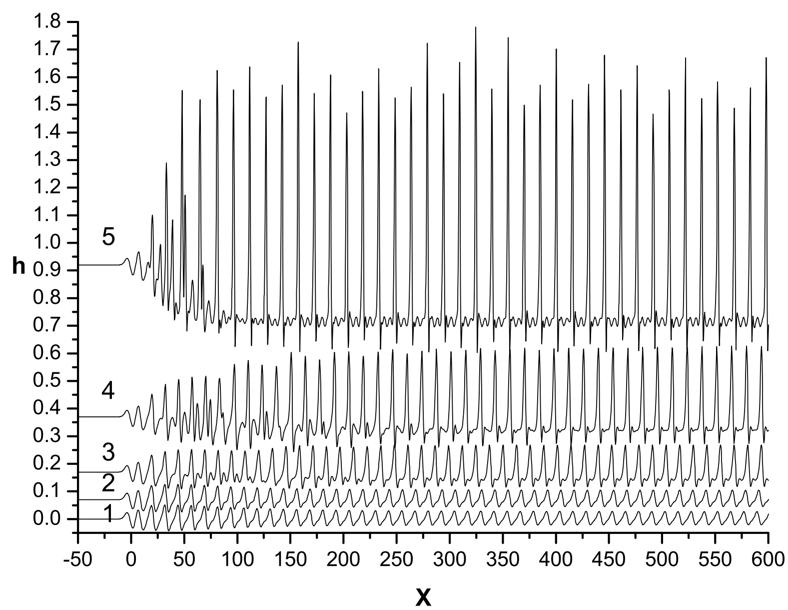

FIG. 5: Flat wall, $\beta = 90°$; surface perturbations for $\omega = 1$ and Re $= 1.967$ and $T = 600$: (1) De $= 0$ (Newtonian), (2) De $= 0.01$, (3) De $= 0.05$, (4) De $= 0.1$, (5) De $= 0.2$.

stabilize by spatial resonance. As shown in the right side of Fig. 8, all viscoelastic flows are stabilized completely by spatial resonance when $L = 6$, which is larger than before too.

The results for $\omega = 2$ and Re $= 2.783$ and flow down a flat wall are given in Fig. 9. Here, the results for De $= 0.2$ have a wave-number which might be violating the small wave-number approximation assumed from the onset. However, the results are presented to show that even those very unstable conditions can be stabilized by spatial resonance. As shown on the left side of Fig. 10, the Newtonian and the De $= 0.01$ fluids are stabilized in the space interval when

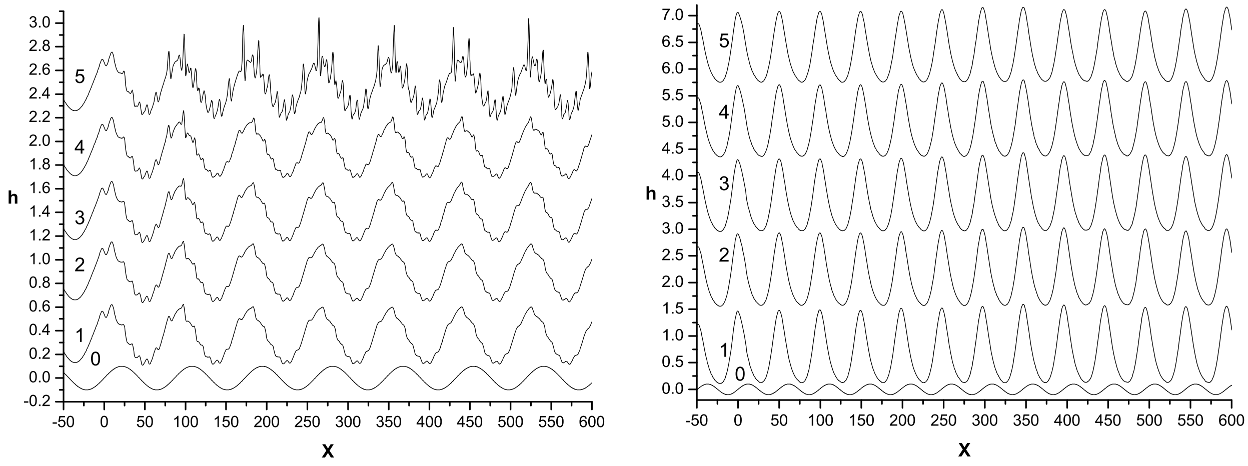

FIG. 6: Wavy wall, left $L = 7$ and right $L = 4$, $\beta = 90°$; surface perturbations for $\omega = 1$ and Re = 1.967 and $T = 600$: (0) wall, (1) De = 0 (Newtonian), (2) De = 0.01, (3) De = 0.05, (4) De = 0.1, (5) De = 0.2.

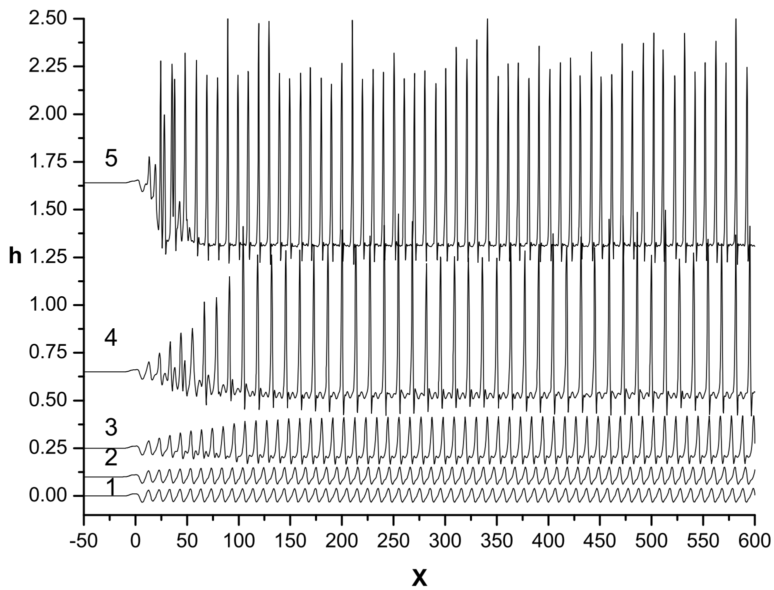

FIG. 7: Flat wall, $\beta = 90°$; surface perturbations for ω = 1.5 and Re = 2.41 and $T = 600$: (1) De = 0 (Newtonian), (2) De = 0.01, (3) De = 0.05, (4) De = 0.1, (5) De = 0.2.

$L = 10$, larger than before. On the right side of Fig. 10 it is seen that for $L = 6$ the perturbations are strongly stabilized by spatial resonance. Note that the perturbations cannot survive one half of a wave of the response of the film.

4. CONCLUSIONS

The results presented above show the important influence the vertical wall waviness may have on the instability of a viscoelastic thin film. The viscoelastic parameter, the Deborah number, destabilizes the flow and becomes more

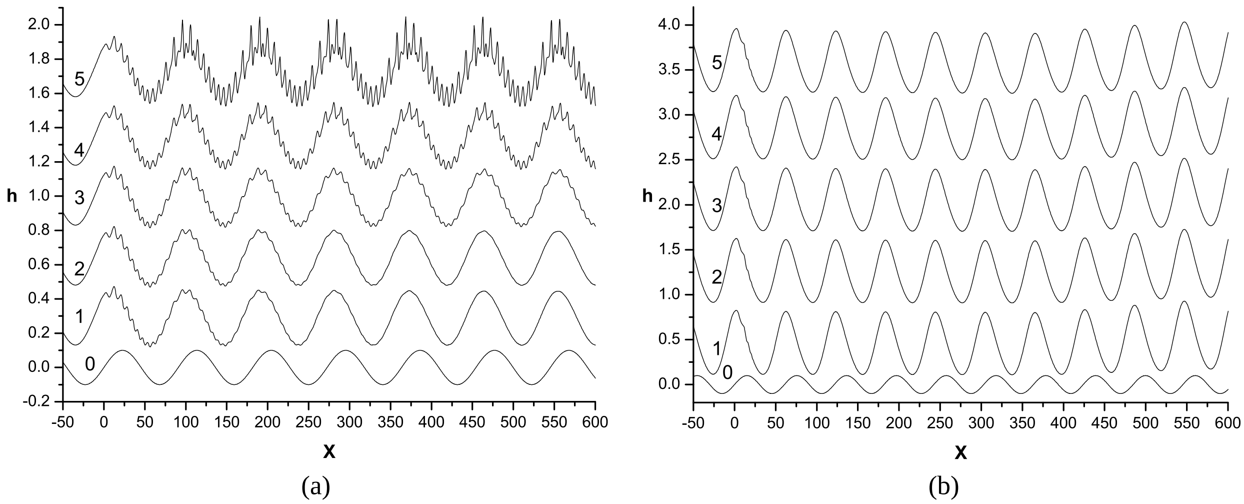

FIG. 8: Wavy wall, left $L = 9$ and right $L = 6$, $\beta = 90°$; surface perturbations for $\omega = 1.5$ and Re $= 2.41$ and $T = 600$: (0) wall, (1) De $= 0$ (Newtonian), (2) De $= 0.01$, (3) De $= 0.05$, (4) De $= 0.1$, (5) De $= 0.2$.

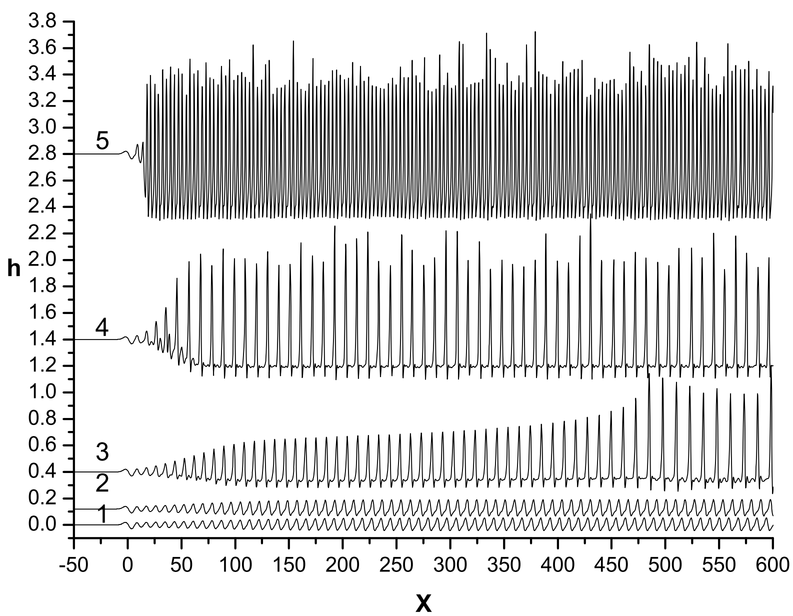

FIG. 9: Flat wall, $\beta = 90°$; surface perturbations for $\omega = 2$ and Re $= 2.783$ and $T = 600$: (1) De $= 0$ (Newtonian), (2) De $= 0.01$, (3) De $= 0.05$, (4) De $= 0.1$, (5) De $= 0.2$.

important when the Reynolds number increases because the viscoelastic term in Eq. (13) is multiplied by Re^2. The numerical solutions for a viscoelastic film flowing down a flat wall are the evidence of that influence. It is interesting to note that Eq. (13) presents no wall deformation effects in the viscoelastic term. At first, it was thought that this might be an obstacle to reach the goal to stabilize passively the viscoelastic film perturbations. However, it is shown in the small wave-number approximation that by means of an adequate wall wavelength it is possible to stabilize the perturbations in a range of viscoelastic parameters and Reynolds numbers. It is found that flows near to their curves

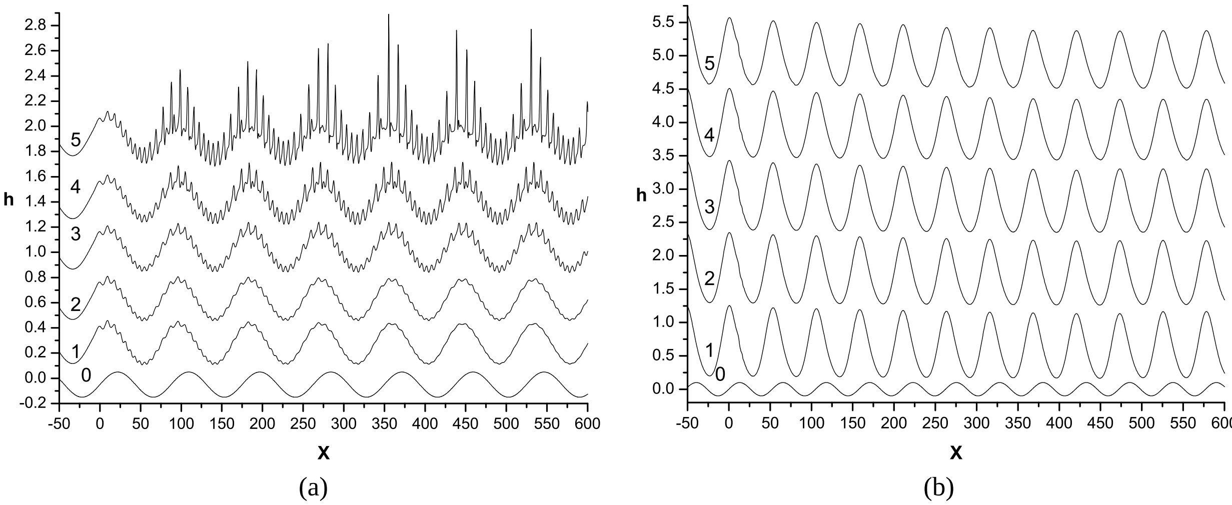

FIG. 10: Wavy wall, left $L = 10$ and right $L = 6$. $\beta = 90°$; surface perturbations for $\omega = 2$ and Re $= 2.783$ and $T = 600$: (0) wall, (1) De $= 0$ (Newtonian), (2) De $= 0.01$, (3) De $= 0.05$, (4) De $= 0.1$, (5) De $= 0.2$.

of subcriticality are more difficult to stabilize by spatial resonance. This difficulty is reflected in the magnitude of the L needed for resonance. Numerical results near to subcriticality show that L has to be lowered until $L = 4$ for optimal resonance effects. For large Re the perturbations are easier to stabilize with a larger L and for some flows $L = 10$ is small enough. It is important to note that the perturbations can be stabilized for the full range of De investigated when $L = 4$.

Therefore, the conclusions are that in the small wave-number approximation it is possible to stabilize the perturbations propagating on the free surface of a viscoelastic film flowing down a wavy wall. The flow is stabilized in a passive way by means of spatial resonance occurring in a relation between the wall and free surface perturbations wavelengths.

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