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The Pauli Exclusion Principle. Can It Be Proved?

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Abstract The modern state of the Pauli exclusion principle studies is discussed. The Pauli exclusion principle can be considered from two viewpoints. On the one hand, it asserts that particles with half-integer spin (fermions) are described by antisymmetric wave functions, and particles with integer spin (bosons) are described by symmetric wave functions. This is a so-called spin-statistics connection. The reasons why the spin-statistics connection exists are still unknown, see discussion in text. On the other hand, according to the Pauli exclusion principle, the permutation symmetry of the total wave functions can be only of two types: symmetric or antisymmetric, all other types of permutation symmetry are forbidden; although the solutions of the Schrödinger equation may belong to any representation of the permutation group, including the multi-dimensional ones. It is demonstrated that the proofs of the Pauli exclusion principle in some textbooks on quantum mechanics are incorrect and, in general, the indistinguishability principle is insensitive to the permutation symmetry of the wave function and cannot be used as a criterion for the verification of the Pauli exclusion principle. Heuristic arguments are given in favor that the existence in nature only the one-dimensional permutation representations (symmetric and antisymmetric) are not accidental. As follows from the analysis of possible scenarios, the permission of multi-dimensional representations of the permutation group leads to contradictions with the concept of particle identity and their independence. Thus, the prohibition of the degenerate permutation states by the Pauli exclusion principle follows from the general physical assumptions underlying quantum theory.

Keywords Pauli exclusion principle · Spin-statistics connection · Indistinguishability principle · Permutation symmetry

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1 Introduction

Wolfgang Pauli formulated his principle before the creation of the contemporary quantum mechanics (1925–1927). He arrived at formulation of this principle trying to explain the regularities in the classification of atomic spectral terms in a strong magnetic field. In paper submitted for publication in January 1925, Pauli formulated his principle as follows [1]:

> "*In an atom there cannot be two or more equivalent electrons for which the values of all four quantum numbers coincide. If an electron exists in an atom for which all of these numbers have definite values, then this state is* 'occupied'."

At that time, the fourth quantum number was not described by any model. Trying to describe the doublet structure in the spectra of alkaline atoms Pauli called the property associated with it as the "characteristic two-valuedness of the quantum properties of the electron which cannot be described classically" [2].

This non-classical two-valued nature of electron is now called *spin*. In anticipating the quantum nature of the magnetic moment of electron before the creation of quantum mechanics, Pauli exhibited a striking intuition. It is interesting to note that this intuition together with Pauli's inherent rigor of thought did not allow him at once to admit the hypothesis of the spin explaining the doublet splitting in the spectra of alkali atoms, which had been proposed by Kronig (who did not published it) and independently by Uhlenbeck and Goudsmit [3]. Pauli's objections resulted from the fact that their spin hypothesis was based on the classical concept of rotation of the electron about its own axis. Upon meeting with Bohr who had fallen under the influence of the explanation of the doublet splitting into the favor on the rotating-electron hypothesis, Pauli expressed the regret that a new "heresy" had arisen in atomic physics, as van der Waerden wrote in his recollections [4].

It is now clear that Pauli was right in not agreeing with the classical interpretation of the fourth degree of freedom. The spin cannot in principle be described by classical physics. The first studies devoted to applying the newborn quantum mechanics to many-particle systems were performed by Heisenberg [5] and Dirac [6]. In these studies, the Pauli principle, formulated as the prohibition for two electrons to occupy the same quantum state, was derived as consequence of the antisymmetry of the wave function of the system of electrons. Dirac [6] came to the conclusion that the light quanta must be described by the symmetric wave functions. He specially noted that a system of electrons cannot be described by the symmetric wave functions since the latter allow any number of electrons to occupy a quantum state.

Thus, with the creation of quantum mechanics, the prohibition on the occupation numbers of electron system states was supplemented by the prohibition of all types of permutation symmetry of electron wave functions except for antisymmetric one. Later on, an analysis of experimental data has permitted to formulate the Pauli exclusion principle for all known elementary particles. Namely:

> *The only possible states of a system of identical particles possessing spin s are those for which the total wave function transforms upon interchange of any two particles as*

$$P_{ij}\Psi(1,\ldots,i,\ldots j,\ldots,N) = (-1)^{2s}\Psi(1,\ldots,i,\ldots j,\ldots,N), \quad (1)$$

that is, it is symmetric for integer values of s (the Bose-Einstein statistics) and antisymmetric for half-integer s (the Fermi-Dirac statistics).

The Pauli exclusion principle also holds for the permutation symmetry of composite-particle wave functions, e.g. for nuclei. The latter consist of nucleons: protons and neutrons which are fermions because have $s = 1/2$. Depending on the value of the total nuclear spin, one can speak of boson nuclei and fermion nuclei. The nuclei with even number of nucleons have an integer value of the total spin and belong to the boson particles. The nuclei with odd number of nucleons have a half-integer value of the total spin and belong to the fermion particles.

The well-known example is the $^{16}O_2$ molecule. The nucleus ^{16}O consists of even number of nucleons and has the total nuclear spin $I = 0$, hence it is a boson composite particle; so the total wave function of the $^{16}O_2$ molecule must be symmetric under the permutations of nuclei. At the Born-Oppenheimer approximation the molecular wave function can be represented as a product of the electronic, Ψ_{el} and nuclear, Φ_n, wave functions. At the equilibrium distances the nuclear wave function, in its turn, can be represented as a product of the vibrational, Φ_{vib}, and rotational, Φ_{rot}, wave functions. Thus,

$$\Psi\left(^{16}O_a - {}^{16}O_b\right) = \Psi_{el}(ab)\Phi_{vib}(ab)\Phi_{rot}(ab). \tag{2}$$

We did not write the nuclear spin function, because the nucleus ^{16}O has zero spin. The vibrational wave function, $\Phi_{vib}(ab)$, depends only on the magnitude of the interatomic distance and remains unaltered under the interchange of the nuclei. The ground state electronic wave, $\Psi_{el}(ab)$, is antisymmetric under the interchange of the nuclei. Hence, for fulfilling the boson symmetry of the total wave function (2) the rotational wave function, $\Phi_{rot}(ab)$, must be also antisymmetric under the interchange of the nuclei. The symmetry of the rotational wave function in the state with the rotational angular moment K is determined by the factor $(-1)^K$. Therefore, in the ground electronic state the even values of K are forbidden and only odd values of K are allowed. Just this consequence from the Pauli exclusion principle was revealed in the earlier spectroscopic measurements in 1927 [7] and now is well established. The test of possible violation of the Pauli exclusion principle for the $^{16}O_2$ molecule [8] gave the probability of such violation $\le 8 \times 10^{-7}$.

The group-theoretical procedure for finding allowed by the Pauli exclusion principle quantum states for an arbitrary system composed of many particle subsystems was elaborated in Refs. [9, 10], see also Ref. [11], Chap. 6. It can be atoms in molecules, complexes of impurity centers in crystals and so forth. Depending upon the value of its total spin, the subsystems behave under permutations as fermion or boson composite particles.

For elementary particles, the permutation symmetry of wave function is directly connected with its statistics. It is worth-while to note that the situation becomes complicated in the case of composite particles. Although, the wave function of composite particles can be characterized only by the boson or fermion permutation symmetry, its second quantization operators do not obey the pure boson or fermion commutation relations [12–14], see also recent publications [15, 16]. When the internal structure of the composite particle is taken into account, the deviations from the purely bosonic or fermionic properties are usually appeared. For two fermions it was revealed long ago in the BCS theory of superconductivity based on the conception of the Cooper pairs, see Ref. [17]. The direct calculation shows that the commutation relation of operators of creation, $b^{+}_{\mathbf{k}}$, and annihilation, $b_{\mathbf{k}}$, of Cooper's pair, where $\mathbf{k}$ is the electron impulse, are bosonic for $\mathbf{k} \neq \mathbf{k}'$. But for $\mathbf{k} = \mathbf{k}'$ their commutation relations differ from the boson commutations relations and the occupation numbers are exactly that the Pauli exclusion principle demands for fermions.

To the best of our knowledge, the first studies of the effective repulsion between composite particles consisting of identical fermions were performed by Zeldovich [18] who showed that the Pauli repulsion arises when the overlap of wave functions became appreciable. This leads to the well known in atomic and molecular physics *exchange interaction* stipulated by the requirement of antisymmetry of many-electron wave functions. Hence, the exchange interaction is a consequence of the Pauli exclusion principle. At present different computational schemes are elaborated, in which the Pauli repulsion operator [19, 20], the so-called Pauli blockade method [21, 22], and some other approaches are used.

In Refs. [16, 23] the Pauli exclusion principle is connected with such interesting and mysterious quantum phenomenon as *entanglement* [24], which at present is broadly implemented in quantum information theory [25]. The term "entanglement" was introduced by Schrödinger [26] when he analyzed the so-called Einstein-Podolsky-Rosen paradox [27], see also Refs. [28–30].

All experimental data known to date agree with the Pauli exclusion principle. A detailed discussion of the theory and experiments on the search for possible small violations of the Pauli exclusion principle can be found in Proceedings [31] and reviews [32–34]. At present the systematic experimental study of the validity of the Pauli exclusion principle for electrons is carrying out by the VIP collaboration [35]. In their experiment they perform a search of X-rays from the Pauli-forbidden atomic transition from the $2p$ shell to the closed $1s^2$ shell of Cu atoms. The obtained probability that the Pauli exclusion principle is violated, according to their measurements [36, 37] is

$$\frac{1}{2}\beta^2 < 6 \times 10^{-29}. \tag{3}$$

In the experiments performed in the Los Alamos laboratory by Elliott et al. [38] Pb instead of Cu was used.. They reported a much stronger limit on the violation of the Pauli exclusion principle for electrons. Namely:

$$\frac{1}{2}\beta^2 < 2.6 \times 10^{-39}. \tag{4}$$

It must be mentioned that this limit was obtained by a modified method of the processing of the experimental data. As noted in Ref. [38], in the conductor there are two kinds of electrons: the current electrons that have no previous contacts with the target and the electrons within the target, which are "less new". The authors [38] took into account *all* free electrons. The application of this approach to the VIP data also changes their limit on ten orders. On the other hand, it seems that the processing method used in Ref. [38] cannot be rigorously based. In any case, as follows from experimental data, the probability of the non-Pauli states, e.g. $(ns)^3$, is practically zero.

It is worth-while to make one additional comment in connection with the experimental studies. Usually experimenters consider the violation of the Pauli exclusion principle as a small admixture of the symmetric wave functions to the antisymmetric state. They start from the Ignatiev-Kuzmin [39] and Greenberg-Mohapatra theories [32, 40]. These theories are based on the second quantization field formalism, in which only the symmetric and antisymmetric states are defined. In general, this limitation on the symmetry of the states is not valid, because the solutions of the Schrödinger equation may belong to any representation of the permutation group. If the Pauli exclusion principle is violated, the electron system can be in the state with an arbitrary permutation symmetry and not only in the symmetric state, see Sects. 2 and 3.

Thus the Pauli exclusion principle is based on the analysis of experimental data. Pauli himself was never satisfied by this. In his Nobel Prize lecture Pauli said [41]:

> "*Already in my initial paper, I especially emphasized the fact that I could not find a logical substantiation for the exclusion principle nor derive it from more general assumptions. I always had a feeling, which remains until this day, that this is the fault of some flaw in the theory*."

Let us stress that this was said in 1946, after the Pauli famous theorem [42] of the relation between spin and statistics. In this theorem, Pauli did not give a direct proof. He showed that due to some physical contradictions, the second quantization operators for particles with integral spins cannot obey the fermion commutation relations; while for particles with half-integral spins, their second quantization operators cannot obey the boson commutation relations. From this Pauli concluded that particles with integral spin have to obey the Bose-Einstein statistics, while those with half-integral spin have to obey the Fermi-Dirac statistics.

Thus, according to the Pauli theorem, the connection between the value of spin and the permutation symmetry of many-particle wave function, Eq. (1), follows if we assume that particles can obey only two types of commutation relations: boson or fermion relations. At that time it was believed that it is really so. However in 1953, Green [43] (and independently Volkov [44]) showed that the more general, paraboson and parafermion trilinear commutation relations, satisfying all physical requirements and containing the boson and fermion commutation relations as particular cases, can be introduced. A corresponding parastatistics is classified by its rank p. For the parafermi statistics p is the maximum occupation number. For $p = 1$ the parafermi statistics becomes identical to the Fermi-Dirac statistics. For the parabose statistics there are no restrictions in the occupation numbers; for $p = 1$ the parabose statistics is reduced to the Bose-Einstein statistics (for more details see book by Ohnuki and Kamefuchi [45]).

Since, there are no prohibitions on the existence of elementary particles obeying the parastatistics commutation relations; the proof of the spin-statistical theorem [42] loses its base. After 1940, numerous proofs of the spin-statistics theorem were published. All these proofs contain some explicit (or implicit) assumptions, see book by Duck and Sudarshan [46], review on it by Wightman [47], and Proceedings [31]. As emphasized by Berry and Robbins [48], the relation between spin and statistics "cries out for understanding".

Up to date the elementary particles obeying the parastatistics are not detected. Although, as discussed in Refs. [49–51], the ordinary fermions, which differ by some internal quantum numbers but are similar dynamically, can be described by the parafermi statistics. In this case, fermions with *different* internal quantum numbers are considered as *non-identical distinguishable* particles. So, *quarks* with 3 colors obey the parafermi statistics of rank $p = 3$; *nucleons* in nuclei (isotope spin$\frac{1}{2}$) obey the parafermi statistics of rank $p = 2$. It is important to stress that the parafermi statistics of rank p describes systems with p different types of fermions and each type obeys the Fermi-Dirac statistics. The total wave function for parafermions always can be constructed as an antisymmetric function in full accordance with the Pauli exclusion principle.

In 1976, the author [52] revealed that the parafermi statistics is realized for quasiparticles in a crystal lattice, e.g. for the Frenkel excitons or magnons, but due to a periodical crystal field, the Green trilinear commutation relations are modified by the quasi-impulse conservation law. Later on, it was shown that the introduced by Kaplan modified parafermi statistics [52] is valid for different types of quasiparticles in a periodical lattice: polaritons [53], defectons [54], the Wannier-Mott excitons [55], delocalized holes in crystals [56], and delocalized coupled hole pairs [57], see also Refs. [58, 59].

The study of properties of the quasiparticles in a periodical lattice revealed [52, 56] that even in the absence of dynamical interactions, the quasiparticle system is characterized by some kinematic interaction depending on the deviation of their statistics from the Bose (Fermi) statistics. This kinematic interaction mixes all states of the quasiparticle band. One cannot define an independent quasiparticle in a definite state. The ideal gas of such quasiparticles does not exist fundamentally. There are also no direct connection between the commutation relations for quasiparticle operators and the permutation symmetry of many-quasiparticle wave functions.

Henceforth, we will discuss only systems of identical particles (no quasiparticles) and focus on the symmetry restrictions of the Pauli exclusion principle. According to it, only two types of permutation symmetry of many-particle wave functions are allowed: symmetric and antisymmetric. Both belong to the one-dimensional representations of the permutation group; all other types of the permutation symmetry are forbidden. However, the Schrödinger equation is invariant under any permutation of identical particles. The Hamiltonian of an identical particle system commutes with the permutation operators,

$$[P, H]_- = 0. \quad (5)$$

From this follows that the solutions of the Schrödinger equation may belong to any representation of the permutation group, including multi-dimensional representations. The question might be asked: *whether this limitation on the solutions of the Schrödinger equation follows from the fundamental principles of quantum mechanics or it is an independent principle*?

In the next sections we will discuss the possible answers on this question developing some ideas of our previous publications [60–62]. In Sect. 2 we discuss why proofs of the Pauli exclusion principle in some textbooks on quantum mechanics are incorrect and demonstrate that the indistinguishability principle cannot be used as a criterion for the verification of the Pauli exclusion principle, because it is insensitive to the permutation symmetry of wave function. In Sect. 3 the study of different scenarios following from the allowance of degenerate permutation representations reveals that the latter leads to some contradictions with the concept of particle identity and their independency from each other. The summary and concluding remarks are given in Sect. 4 and in Appendix the necessary mathematical apparatus of the permutation group is represented.

2 Indistinguishability of Identical Particles and the Symmetry Postulate

There are two view-points on the problem of independency of the Pauli exclusion principle from other fundamental quantum-mechanical postulates. Some physicists, including the founders of quantum mechanics Pauli [63] and Dirac [64] (see also Shiff [65] and Messiah [66]), have assumed that there are no laws in Nature that forbid the existence of particles described by wave functions with more complicated permutation symmetry than those of bosons and fermions, and that the existing limitations are only due to the specific properties of the known elementary particles.

Messiah [66, 67] has even introduced the term *symmetrization postulate* to emphasize the primary nature of the constraint on the allowed types of the wave function permutation symmetry. It should be mentioned that before Messiah, the independence of the Pauli exclusion principle from other fundamental principles of quantum mechanics was stressed by Pauli in his speech [63] when he entered the Princeton Institute for Advance Studies.

In fact, the existence of permutation degeneracy should not introduce additional uncertainty into characteristic of the state. From the Wigner-Eckart theorem generalized for the permutation group, see Eq. (4.60) in book [11], follows that the matrix element of an operator L, which is symmetric in all the particles, can be presented as

$$\langle \Psi_r^{[\lambda]} | \hat{\mathrm{L}} | \Psi_r^{[\lambda]} \rangle = \delta_{r\bar{r}} \langle \Gamma^{[\lambda]} \| \hat{\mathrm{L}} \| \Gamma^{[\lambda]} \rangle, \tag{6}$$

where index r labels the basic functions of the representation $\Gamma^{[\lambda]}$ of the permutation group and $[\lambda]$ is the Young diagram, see Appendix. The double vertical line in the right-hand side of this formula means that the matrix element is independent on the basic function index. Thus, the expectation value of operator L is the same for all functions belonging to the degenerate state described by an arbitrary irreducible representation $\Gamma^{[\lambda]}$ of the permutation group.

According to another view-point, the symmetry postulate is not an independent principle and can be derived from the fundamental principles of quantum mechanics, in particular, from the principle of indistinguishability of identical particles. This idea is represented not only in articles, see critical comments in Refs. [67, 68], but also in textbooks [69–71], including the famous textbook by Landau and Lifshitz [70]. The incorrectness of proof [69] was mentioned by Girardeau [68]; the proofs [69–71] were critically analyzed in my paper [72]; nevertheless, incorrect proofs of the Pauli principle are still appear in current literature, see, for instance, the review [73]. So, it is worth-while to discuss this matter once more.

The typical argumentation (it is the same in all Refs. [69–71, 73]) is the following. From the requirement that the states of a system obtained by permutations of identical particles must all be physically equivalent, one concludes that the transposition of any two identical particles should multiply the wave function only on an insignificant phase factor,

$$P_{12}\Psi(x_1,x_2)=\Psi(x_2,x_1)=e^{i\alpha}\Psi(x_1,x_2), \tag{7}$$

where α is a real constant and x is the set of spatial and spin variables. One more application of the permutation operator P_{12} gives

$$\Psi(x_1,x_2)=e^{i2\alpha}\Psi(x_1,x_2), \tag{8}$$

or

$$e^{2i\alpha}=1 \quad \text{and} \quad e^{i\alpha}=\pm 1. \tag{9}$$

Since all particles are assumed to be identical, the wave function should change in exactly the same way under transposition of any pair of particles, i.e. it should be either totally symmetric or totally antisymmetric.

This proof contains two essential incorrectness's at once. The first is simply follows from the group theory. Namely: Eq. (7) is valid only for the one-dimensional representations. The application of a group operation to one of basic functions, belonging to some multi-dimensional representation, transforms it in a linear combination of basic functions. Namely,

$$P_{12}\Psi_i=\sum_k \Gamma_{ki}(P_{12})\Psi_k. \tag{10}$$

The application of the permutation operator P_{12} to both sides of Eq. (10) leads to the identity:

$$\begin{aligned} P_{12}\{P_{12}\Psi_i\}=\Psi_i=P_{12}\sum_k \Gamma_{ki}(P_{12})\Psi_k &= \sum_l \Big[\sum_k \Gamma_{lk}(P_{12})\Gamma_{ki}(P_{12})\Big]\Psi_l \\ &= \sum_l \Gamma_{\mathrm{li}}\big(P_{12}^2=I\big)\Psi_l=\Psi_i. \end{aligned} \tag{11}$$

Using this identity we cannot arrive at any information about the symmetry, in contrary with Eq. (8). By requiring that under permutations the wave function must change by no more than a phase factor, one actually *postulates* that the representation of the permutation group, to which the wave function belongs, is one-dimensional. It follows that the proof in Refs. [69–71, 73] is based on the initial statement, which is proved then as a final result.

The second incorrectness in the proof above follows from physical considerations. This proof is directly related to the behavior of the wave function. However, since the wave function is not an observable, the indistinguishability principle is related to it only indirectly via the expressions of measurable quantities. Since in quantum mechanics, the physical quantities are expressed as bilinear forms of wave functions, the indistinguishability principle requires the invariance of these bilinear forms and can be formulated as:

$$\langle P\Psi|\hat{L}|P\Psi\rangle=\langle\Psi|\hat{L}|\Psi\rangle, \tag{12}$$

where $\hat{L}$ is an arbitrary operator. Often, one limits oneself to the requirement that the probability of a given configuration of a system of identical particles must be invariant under permutations [68, 74],

$$P\left|\Psi(x_1,\ldots,x_N)\right|^2 = \left|\Psi(x_1,\ldots,x_N)\right|^2. \tag{13}$$

For a function to satisfy Eq. (13), it is sufficient that under permutations it would change as

$$P\Psi(x_1,\ldots,x_N) = e^{i\alpha_p(x_1,\ldots,x_N)}\Psi(x_1,\ldots,x_N), \tag{14}$$

i.e. unlike the requirement of condition (7), in the general case the phase is a function of coordinates and the permutation, and Eq. (8) evidently does not hold.

Most other proofs of the symmetry postulate contain unjustified constraints. A critical survey of such proofs can be found in Refs. [68, 72]. Proofs of the symmetry postulate without imposing additional constraints have been given by Girardeau [68, 74], who based it on Eq. (13), and in my paper [72] where it was based on Eq. (12). As was noted later by the author [60, 61] these proofs, basing on the indistinguishability principle in the forms (12) and (13), are incorrect, because these equations are correct only for non-degenerate states. In a degenerate state, the system can be described with the equal probability by any one of the basic vectors of the degenerate state. As a result, we can no longer select a pure state (the one that is described by the wave function) and should regard a degenerate state as a mixed one, where each basis vector enters with the same probability [75]. Thus, we must sum both sides of Eqs. (12) and (13) over all wave functions that belong to the degenerate state. For instance, the probability density, which described via the diagonal element of the density matrix, in the case of a degenerate state has the form

$$D_t^{[\lambda]}(x_1,\ldots,x_N;x_1,\ldots,x_N) = \frac{1}{f_\lambda}\sum_{r=1}^{f_\lambda}\Psi_{rt}^{[\lambda]}(x_1,\ldots,x_N)^*\Psi_{rt}^{[\lambda]}(x_1,\ldots,x_N), \tag{15}$$

where the expression (15) is written for the case of the f_λ-dimensional representation $\Gamma^{[\lambda]}$ of the permutation group $\boldsymbol{\pi}_N$ and wave functions $\Psi_{rt}^{[\lambda]}$ are constructed by the Young operators $\omega_{rt}^{[\lambda]}$, see Eq. (33) in Appendix. The possibility of expressing the density matrix through only one of the functions implies that the degeneracy with respect to permutations has been eliminated. However, the latter cannot be achieved without violating the identity of the particles.

Recently S. Zagoulaev (St. Petersburg University) informed me that in 1937 V. Fock presented a proof of the Pauli exclusion principle in his unpublished lectures on quantum mechanics [76]. In his proof Fock substituted the correct expression (14) in equation for an arbitrary operator (12) and applied the variational theorem. As we discussed above, Eq. (12) is valid only in the case of non-degenerate states. The Fock proof is failed if one applies it to the expression valid for degenerate states.

It is not difficult to prove that for every representation $\Gamma^{[\lambda]}$ of the permutation group $\boldsymbol{\pi}_N$, the probability density, Eq. (15), is a group invariant. In the general case it was proved in Ref. [77]. Below I represent this proof for the permutation group.

Let us apply some permutation $P \subset \boldsymbol{\pi}_N$ to the expression (15)

$$PD_t^{[\lambda]} = \frac{1}{f_\lambda}\sum_r\left[\sum_u \Gamma_{ur}^{[\lambda]}(P)^*\Psi_{ut}^{[\lambda]*}\sum_{u'}\Gamma_{u'r}^{[\lambda]}(P)\Psi_{u't}^{[\lambda]}\right]$$
$$= \frac{1}{f_\lambda}\sum_{u,u'}\left(\sum_r \Gamma_{ur}^{[\lambda]}(P)\Gamma_{u'r}^{[\lambda]}(P)\right)\Psi_{ut}^{[\lambda]*}\Psi_{u't}^{[\lambda]}.$$

Due to the orthogonality relations for the matrix elements of irreducible representations, the sum over r is equal to $\delta_{uu'}$, and we arrive at the final result:

$$PD_t^{[\lambda]} = \frac{1}{f_\lambda}\sum_u\left|\Psi_{ut}^{[\lambda]}\right|^2 = D_t^{[\lambda]}. \tag{16}$$

This means that for all irreducible representations $\Gamma^{[\lambda]}$, characterizing the quantum states, the diagonal element of the full density matrix (and all reduced densities matrices as well) transforms according to the totally symmetric one-dimensional representation of $\boldsymbol{\pi}_N$ and in this respect one cannot distinguish between degenerate and nondegenerate states. Thus, the diagonal element of the density matrix is a group invariant.

From this follows that the probability density obeys the indistinguishability principle even in the case of multi-dimensional representations of the permutation group. Thus, the indistinguishability principle is insensitive to the symmetry of wave function and cannot be used as a criterion for selecting the correct symmetry.

Although the Pauli exclusion principle cannot be rigorously derived from other quantum-mechanical postulates, there are some heuristic arguments indicating that the description of an identical particle system by the multi-dimensional representations of the permutation group leads to some contradictions with the concept of the particle identity and their independency. In next section, we discuss these arguments in detail.

3 Some Contradictions with the Concept of Particle Identity and Their Independence in the Degenerate Permutations States

Let us consider a quantum system with the arbitrary number of identical elementary particles without the restrictions imposed by the Pauli exclusion principle. The states of a system of identical particles with the number of particles not conserved can be presented as vectors in the Fock space **F** [78]. The latter is a direct sum of spaces $\mathbf{F}^{(N)}$ corresponding to a fixed number of particles N

$$\mathbf{F} \doteq \sum_{N=0}^{\infty}\mathbf{F}^{(N)}. \tag{17}$$

Each of the space $\mathbf{F}^{(N)}$ can be presented as a direct product of one-particle spaces **f**:

$$\mathbf{F}^{(N)} = \underbrace{\mathbf{f}\otimes\mathbf{f}\otimes\cdots\otimes\mathbf{f}}_{N}. \tag{18}$$

The basic vectors of $\mathbf{F}^{(N)}$ are the product of one-particle vectors $|v_k(k)\rangle$ belonging to spaces **f**; k in the parenthesis denotes the set of particle spin and space coordinates,

$$\left|\xi^{(N)}\right\rangle = \left|v_1(1)\right\rangle\left|v_2(2)\right\rangle\cdots\left|v_N(N)\right\rangle. \tag{19}$$

For simplicity, let us consider the case where all one-particle vectors in Eq. (19) are different. There will be no qualitative changes in the results, if some of the vectors coincide. $|v_k(k)\rangle$ are spin-orbitals, on which the total wave function is constructed.

One can produce $N!$ new many-particle vectors by applying to the many-particle vector (19) $N!$ permutations of the particle coordinates. These new vectors also belong to $\mathbf{F}^{(N)}$ and form in it a certain invariant subspace which is reducible. The $N!$ basic vectors of the latter, $P|\xi^{(N)}\rangle$, make up the regular representation of the permutation group $\boldsymbol{\pi}_N$. As is known in the group theory, the regular representation is decomposed into irreducible representations, each of which appears a number of times equal to its dimension. The space $\varepsilon^{(N)}$ falls into the direct sum

$$\varepsilon_\xi^{(N)} \dot{=} \sum_{\lambda_N} f_{\lambda_N} \varepsilon_\xi^{[\lambda_N]}, \tag{20}$$

where $\varepsilon_\xi^{[\lambda_N]}$ is an irreducible subspace of dimension f_λ drawn over the basic vectors $|[\lambda_N]r\rangle$, and λ_N is a Young diagram with N boxes. The basic vectors $|[\lambda_N]r\rangle$ can be constructed of non-symmetrized basic vector $|\xi^{(N)}\rangle$ by using the Young operators $\omega_{rt}^{[\lambda_N]}$, see Appendix,

$$\big|[\lambda_N]rt\big\rangle = \omega_{rt}^{[\lambda_N]}\big|\xi^N\big\rangle = \left(\frac{f_\lambda}{N!}\right)^{\frac{1}{2}} \sum_P \Gamma_{rt}^{[\lambda_N]}(P) P\big|\xi^{(N)}\big\rangle, \tag{21}$$

where $\Gamma_{rt}^{[\lambda_N]}(P)$ are the matrix elements of representation $\Gamma^{[\lambda_N]}$ and index t distinguishes between the bases in accordance with the decomposition of $\varepsilon_\xi^{(N)}$ into f_λ invariant subspaces and describes the symmetry under permutations of the particle vector indices.

Thus, a space with a fixed number of particles can always be divided into irreducible subspaces $\varepsilon_\xi^{[\lambda_N]}$, each of which is characterized by a certain permutation symmetry given by a Young diagram with N boxes. The symmetry postulate demands that the basis vectors of a system of N identical particles belong to one of the two subspaces characterized by irreducible one-dimensional representations, either $[N]$ or $[1^N]$. All other subspaces are "empty". Let us examine the situation that arises when no symmetry constraints are imposed and consider the system of N identical particles described by basic vectors belonging to some irreducible subspace $\varepsilon_\xi^{[\lambda_N]}$.

One of the consequences of the different permutation symmetry of state vectors for bosons and fermions is the dependence of the energy of the system on the particle statistics. For the same law of dynamic interaction, the so-called exchange terms enter the expression for the energy of fermion and boson system with opposite signs. Let us calculate the energy of the multi-dimensional permutation state $|[\lambda]rt\rangle$. The energy of the system in a degenerate state is

$$E = \mathrm{Tr}(HD), \tag{22}$$

where D is the density operator defined, similarly to Eq. (15), as

$$D_t = \frac{1}{f_\lambda} \sum_{r=1}^{f_\lambda} \big|[\lambda]rt\big\rangle\big\langle[\lambda]rt\big|. \tag{23}$$

The calculation of the trace over the functions with symmetry $[\lambda_N]$ yields

$$E_t^{[\lambda]} = \frac{1}{f_\lambda}\sum_{r=1}^{f_\lambda}\big\langle[\lambda]rt\big|H\big|[\lambda]rt\big\rangle. \tag{24}$$

The matrix element in Eq. (24) has been calculated in Ref. [79] in a general case of non-orthogonal one-particle vectors. In the case where all vectors in Eq. (19) are different and orthogonal one gets

$$E_t^{[\lambda]} = \sum_a \langle v_a|h|v_a\rangle + \sum_{a<b}\big[\langle v_a v_b|g|v_a v_b\rangle + \Gamma_{tt}^{[\lambda]}(P_{ab})\langle v_a v_b|g|v_b v_a\rangle\big], \tag{25}$$

where $\Gamma_{tt}^{[\lambda]}(P_{ab})$ is the diagonal matrix element of the transposition P_{ab} of vectors $|v_a\rangle$ and $|v_b\rangle$ in the right-hand part of Eq. (19); h and g are one- and two-particle interaction operators, respectively. Only exchange terms in Eq. (25) depend upon the symmetry of the state. For one-dimensional representations, $\Gamma_{tt}^{[\lambda]}(P_{ab})$ does not depend on the number of particles and the permutation. $P_{ab} : \Gamma^{[N]}(P_{ab}) = 1$ and $\Gamma^{[1^N]}(P_{ab}) = -1$ for all P_{ab} and N. For multi-dimensional representations, the matrix elements $\Gamma_{tt}^{[\lambda]}(P_{ab})$ depend on $[\ \lambda\ ]$ and P_{ab}; in general, *they a*re different for different pairs of identical particles.[1]

It is natural that a different permutation symmetry of the state vector leads to different expressions for the energy. Taking into account that the transitions between states with different symmetry $[\ \lambda_N]$ are strictly forbidden and each state of N particle system with different $[\ \lambda_N]$ has a different analytical formula for its energy, we must conclude that each type of symmetry $[\ \lambda_N]$ corresponds to a certain kind of particles with statistics determined by this permutation symmetry. On the other hand, the classification of state with respect to the Young diagrams $[\ \lambda_N]$ is connected exclusively with identity of particles. Therefore, it must be some additional *inherent particle characteristics*, which establishes for the N particle system to be in a state with definite permutation symmetry, like integer and half-integer values of particle spin for bosons and fermions, and this inherent characteristic has to be different for different $[\lambda_N]$. So, the particles belonging to the different types of permutation symmetry $[\ \lambda_N]$ are not identical, as it is in the particular cases of bosons, $[N]$, and fermions $[1^N]$.

Let us trace down the genealogy of irreducible subspaces $\varepsilon_\xi^{[\lambda_N]}$. In Fig. 1, the genealogy for all irreducible subspaces with $N = 2$ to 4 is presented.

We called the hypothetical particles characterized by the multi-dimensional representations of the permutation group as *intermedions* implying that they obey some intermediate between fermion and boson statistics. For bosons and fermions there are two non-intersecting chains of irreducible representations: $[N] \to [N+1]$ and $[1^N] \to [1^{N+1}]$, respectively; and the energy expression for each type of particles has the same analytical form that does not depend on the number of particles in a system. The situation changes drastically, if we put into consideration the multi-dimensional

[1] The matrices of transpositions for all irreducible representations of groups $\boldsymbol{\pi}_2$–$\boldsymbol{\pi}_6$ are presented in book [11], Appendix 5.

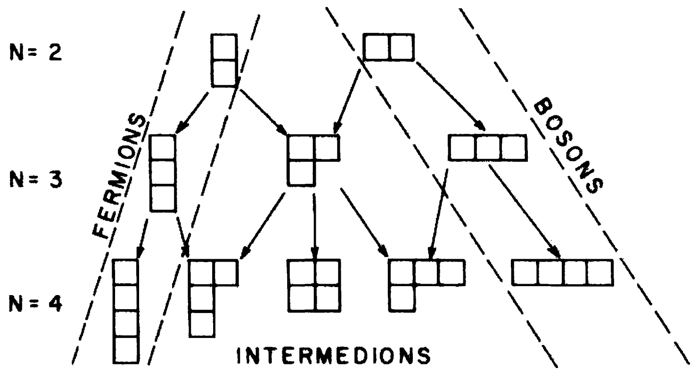

Fig. 1 The Young diagrams for $N = 2-4$ and their genealogy

representations. The number of different statistics depends upon the number of particles in a system and rapidly increases with N. For the multi-dimensional representations we cannot select any non-intersecting chains, as in the fermion and boson cases. According to Fig. 1, the intermedion particles with a definite $[\lambda_N]$ in the Nth generation can originate from particles of different $[\lambda_{N-1}]$ in the $(N-1)$th generation, even from fermions or bosons. The linear combination of wave functions with different permutations symmetry $[\lambda_{N-1}]$ cannot describe the identical particles.

Let us consider $N = 3$. In this case there is only one multi-dimensional representation $[\lambda_3] = [21]$. In the $(N-1)$th generation this representation originates from both $[1^2]$ and [2]. The linear combinations of these representations

$$\Psi_n(x_1, x_2) = c_1\Psi^{[2]}(x_1, x_2) + c_2\Psi^{[1^2]}(x_1, x_2), \tag{26}$$

where x_i denotes three spatial and one spin coordinates of the i-th particle, describes distinguishable particles. In fact,

$$P_{12}\left|\Psi_n(x_1, x_2)\right|^2 = \left|c_1\Psi^{[2]}(x_1, x_2) - c_2\Psi^{[1^2]}(x_1, x_2)\right|^2 \neq \left|\Psi_n(x_1, x_2)\right|^2. \tag{27}$$

The physical picture in which adding one particle changes properties of all particles cannot correspond to a system of *independent* identical particles (although, it cannot be excluded for some quasiparticle systems where we have not an independency of quasiparticles, see the discussion in Introduction). For an ideal gas, it is evident that adding a particle identical to a system of N identical particles cannot change the properties of a new $(N+1)$-particle system. On the other hand, the interaction of identical particles does not change the permutation symmetry of non-interacting particle system. It can be rigorously proved [72]. Namely:

The wave vector $|\Psi\rangle$ of a system of interacting particles characterized by the Hamiltonian

$$H = H_0 + \hat{V} \tag{28}$$

is defined in the same Fock space as the wave vector $|\Psi_0\rangle$ of non-interacting system and it can be generated from the latter by using some unitary transformation

$$|\Psi\rangle = \hat{U}|\Psi_0\rangle. \tag{29}$$

The form of this unitary transformation can be obtained, if we use the well-known relationship that the Brillouin-Wigner perturbation theory [80] is based on:

$$|\Psi\rangle = |\Psi_0\rangle + \frac{Q}{E - H_0}\hat{V}|\Psi\rangle, \tag{30}$$

where Q is the projection operator that projects an arbitrary vector onto the multitude of vectors of Hilbert (Fock) space that are orthogonal to the vector $|\Psi_0\rangle$. Upon substituting Eq. (29) into Eq. (30), we get an equation for $\hat{U}$, from which follows

$$\hat{U}^{-1} = 1 - \frac{Q}{E - H_0}\hat{V}. \tag{31}$$

Since the interaction operator $\hat{V}$ of identical particles is invariant with respect to permutations of identical particles, the operator $\hat{U}$ is also invariant. Hence, according to Eq. (29), the states $|\Psi\rangle$ and $|\Psi_0\rangle$ have the same permutation symmetry.

Thus, the scenario, in which all symmetry types $[\lambda_N]$ are allowed and each of them corresponds to a definite particles statistics, contradicts to the concept of particle identity and their independency from each other.

All contradictions are resolved, if only the one-dimensional irreducible representations of the permutation group are permitted, as follows from the Pauli exclusion principle. Thus, the existence in Nature of only symmetric and antisymmetric types of permutation symmetry is not accidental. It is intimately connected with the identity of particles and their independence.

4 Concluding Remarks

In spite of more than 85 years studies of the Pauli exclusion principle and spin-statistics connection, we still have not a rigorous theoretical ground for it. As we showed in Sect. 2, the indistinguishability principle is insensitive to the permutation symmetry of wave function and is satisfied also by wave functions belonging to the multi-dimensional representations of the permutation group characterized by the Young diagrams $[\lambda_N]$ of general type. So, the indistinguishability principle cannot be used for the verification of the Pauli exclusion principle. Experimental data known to date confirm the Pauli exclusion principle; all elementary particles belong only to one of two statistics: fermion or boson statistics (for quasiparticles, it is not true).

It is demonstrated that the permission for an identical particle system to be in multi-dimensional permutation states leads to contradictions with the concept of particle identity and their independence. Thus, the realization in Nature only one-dimensional permutation states (symmetric and antisymmetric) is by no means accidental. We may not expect that some unknown elementary particles can be described by some multi-dimensional representation of the permutation group.

It is important to stress that the introduction in the parastatistics theory the permutation symmetries, corresponding to the multi-dimensional representations, relates to the wave functions that do not include the internal degrees of freedom. Taking into account the wave functions describing the internal degrees of freedom, we always can obtain the proper permutation symmetry for the total wave function with the full accordance with the Pauli exclusion principle.

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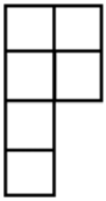
Appendix: Short Necessary Knowledge on the Permutation Group

The permutation symmetry is classified according to the irreducible representations of the permutation group $\boldsymbol{\pi}_N$.[2] The latter are labeled by the Young diagrams

$$[\lambda] = [\lambda_1 \lambda_2 \cdots \lambda_k],$$
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k, \quad \sum_{i=1}^{k} \lambda_i = N, \tag{32}$$

where λ_i is represented by a row of λ_i cells. The presence of several rows of equal length λ_i is convenient to indicate by a power of λ_i. For example,

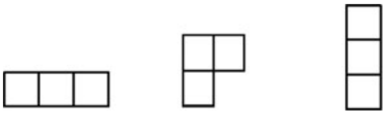


$[\lambda] = [2^2\,1^2]$

It is obvious that one can form from two cells only two Young diagrams:

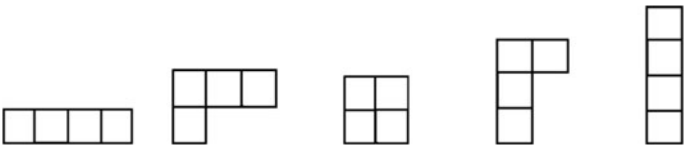
$[2]$ $[1^2]$

For the permutation group of three elements, $\boldsymbol{\pi}_3$, one can form from three cells three Young diagrams:



$[3]$ $[2\ 1]$ $[1^3]$

The group $\boldsymbol{\pi}_4$ has five Young diagrams:



$[4]$ $[3\ 1]$ $[2^2]$ $[2\ 1^2]$ $[1^4]$

Each Young diagram $[\lambda]$ uniquely corresponds to a specific irreducible representation $\Gamma^{[\lambda]}$ of the group $\boldsymbol{\pi}_N$. The assignment of a Young diagram determines the

[2] For a more detailed treatise see books by Kaplan [11] and Hamermesh [81].

permutation symmetry of the basis functions for an irreducible representation, i.e. determines the behavior of the basis functions under permutations of their arguments. A diagram with only one row corresponds to a function symmetrical in all its arguments. A Young diagram with one column corresponds to a completely antisymmetrical function. All other types of diagrams correspond to intermediate types of symmetry. There are certain rules that enable one to find the matrices of irreducible representations of the permutations group from the form of the corresponding Young diagram. Such rules are especially simple in the case of the so-called standard orthogonal representation (this is the Young-Yamanouchi representation; see Ref. [11]).

The basis functions for an irreducible representation $\Gamma^{[\lambda]}$ can be constructed by means of the so-called normalized Young operators [11],[3]

$$\omega_{rt}^{[\lambda]} = \sqrt{\frac{f_\lambda}{N!}} \sum_P \Gamma_{rt}^{[\lambda]}(P) P, \tag{33}$$

where the summation over P runs over all the $N!$ permutations in the group $\boldsymbol{\pi}_N$, $\Gamma_{rt}^{[\lambda]}(P)$ are the matrix elements and f_λ is the dimension of the irreducible representation $\Gamma^{[\lambda]}$. The application of operator (33) to a nonsymmetrized product of orthonormal one-particle functions φ_a

$$\Phi_0 = \varphi_1(1)\varphi_2(2)\cdots\varphi_N(N) \tag{34}$$

produces a normalized function

$$\Phi_{rt}^{\lambda} = \omega_{rt}^{[\lambda]}\Phi_0 = \sqrt{\frac{f_\lambda}{N!}} \sum_P \Gamma_{rt}^{[\lambda]}(P) P\Phi_0 \tag{35}$$

transforming in accordance with the representation $\Gamma^{[\lambda]}$. Let us prove this statement applying an arbitrary permutation Q of the group $\boldsymbol{\pi}_N$ to the function (35):

$$Q\Phi_{rt}^{[\lambda]} = \sqrt{\frac{f_\lambda}{N!}} \sum_P \Gamma_{rt}^{[\lambda]}(P) QP\Phi_0 = \sqrt{\frac{f_\lambda}{N!}} \sum_R \Gamma_{rt}^{[\lambda]}\big(Q^{-1}R\big) R\Phi_0. \tag{36}$$

In this equation we have denoted the permutation QP by R and made use of the invariance properties of a sum over all group elements. Further, we write the matrix element of the product of permutations as products of matrix elements and make use of the property of orthogonal matrices:

$$\Gamma_{rt}^{[\lambda]}\big(Q^{-1}R\big) = \sum_u \Gamma_{ru}^{[\lambda]}\big(Q^{-1}\big)\Gamma_{ut}^{[\lambda]}(R) = \sum_u \Gamma_{ur}^{[\lambda]}(Q)\Gamma_{ut}^{[\lambda]}(R). \tag{37}$$

Substituting (37) in (36), we obtain finally

$$Q\Phi_{rt}^{[\lambda]} = \sqrt{\frac{f_\lambda}{N!}} \sum_u \Gamma_{ur}^{[\lambda]}(Q) \left(\sum_R \Gamma_{ut}^{[\lambda]}(R) R\Phi_0 \right) = \sum_u \Gamma_{ur}^{[\lambda]}(Q)\Phi_{ut}^{[\lambda]}. \tag{38}$$

The function $\Phi_{rt}^{[\lambda]}$ transforms as the r-th column of the irreducible representation $\Gamma^{[\lambda]}$, and the set of f_λ functions $\Phi_{rt}^{[\lambda]}$ with fixed second index t forms a basis for

[3] Operator (33) should not be mixed up with the operator that symmetrizes the rows and antisymmetrizes the columns in Young diagram, which is also often referred to as the Young operator [81].

the irreducible representation $\Gamma^{[\lambda]}$. One can form altogether f_λ independent bases corresponding to the number of different values of t. This should be expected, since $N!$ functions $P\Phi_0$ form a basis for the regular representation of $\boldsymbol{\pi}_N$, and in the decomposition of the regular representation, each irreducible representation occurs as many times as its dimension. The first index, r, characterizes the symmetry of the function $\Phi_{rt}^{[\lambda]}$ under permutation of the arguments. It can be shown [11] that the second index, t, enumerating the different bases of $\Gamma^{[\lambda]}$, characterizes the symmetry of $\Phi_{rt}^{[\lambda]}$ under permutations of the one-particle functions φ_a.

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