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BEC MODEL OF HIGH-T_c SUPERCONDUCTIVITY IN LAYERED CUPRATES

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High-T_c superconductivity in layered cuprates is described in a BCS-BEC formalism with linearly-dispersive s- and d-wave Cooper pairs moving in quasi-2D finite-width layers around the CuO_2 planes. This yields a closed formula for T_c involving the layer width, the Debye frequency, the pairing energy and the in-plane penetration depth. The new formula has no free parameters and reasonably reproduces empirical values of superconducting T_cs for 11 different layered superconductors over a wide doping regime including YBCO itself as well as other compounds like LSCO, BSCCO and TBCCO. In agreement with the London formalism, the formula also yields a fair description of the T_c dependence of the lower critical magnetic field in highly underdoped YBCO.

Keywords: Bose–Einstein condensation; cuprate superconductors; extended BCS theory.

1. Introduction

It appears widely accepted that central to high-T_c cuprate superconductivity (HTSC) is the layered two-dimensional (2D) structure of copper oxides and that superconducting pairing occurs mainly on the CuO_2 planes. However, the precise dynamical nature of the pairing is still the subject of intense research. Recent experiments based on angle-resolved photo emission spectroscopy (ARPES) of underdoped cuprates suggest that bound-fermion Cooper pairs (CPs) form already at and below temperatures higher than the critical transition temperature T_c.[1–5] The bosonic nature of these CPs is consistent with the so-called "Uemura scaling relation"[6–8] of data from muon-spin-relaxation (μSR), neutron and Raman

scattering and ARPES measurements which exhibit T_c versus Fermi temperatures $T_F \equiv E_F/k_B$, where E_F is the Fermi energy associated with single charge carriers, usually holes, and k_B is the Boltzmann constant.

Empirical T_cs of many cuprate as well as other so-called "exotic" superconductors straddle a diagonal line *parallel* but below the Uemura-plot line associated with the simple BEC formula $T_B \simeq 3.31\hbar^2 n_B^{2/3}/m_B k_B \simeq 0.218 T_F$ corresponding to an ideal gas of bosons of mass $m_B = 2m^*$ and number density $n_B = n_s/2$, where m^* is the electron effective mass and n_s the number density of individual charge carriers. The first mentioned "empirical" diagonal line is shifted *down* from T_B by a factor 4–5, a fact judged[9] to be of "fundamental importance of the BEC concept in cuprates." Several theoretical[10–14] papers based on BEC proposed that HTSC might be rooted in a 2D Bose–Einstein condensate (BEC) of CPs pre-existing above T_c and coupled through a BCS-like phonon mechanism,[17,18] originally taken as s-wave. In Refs. 15 and 16 this scenario of the pseudogap is studied in detail.

As apparently first reported by Schrieffer,[19] the Cooper model interaction[20] leads to an approximate *linear* energy-versus-center-of-mass-momentum (CMM) dispersion relation with the leading term $(1/2)v_F\hbar K$ for excited CPs propagating in the Fermi sea, where K is the CMM wavenumber and v_F the Fermi velocity. This linearity makes a Bose–Einstein condensate (BEC) possible in all space dimensions $d > 1$. The formation of BEC of CPs in 2D does not violate Hohenberg's theorem[21] as this theorem holds only for quadratically-dispersive particles. BEC schemes provide a correct description of other relevant physical properties of HTSCs such as a short coherence length, a type II magnetic behavior and the observed temperature dependence of the electronic heat capacity.[10] They also lead to excellent fits[22] of the condensate fraction curves for quasi-2D cuprates just below T_c, as well as for 3D and even quasi-1D SCs. The leading-order linearity is *not* induced by the particular interfermion interaction binding the CPs but is a consequence of the nonzero-v_F Fermi sea in which a CP by definition propagates (see Refs. 14, 23–26 for further discussion).

On the other hand, ARPES data have been interpreted to suggest that HTSC involves an unconventional $d_{x^2-y^2}$ orbital pairing symmetry, since the energy gap Δ (a measure of the energy needed to break a CP) displays a functional dependence $\Delta = \Delta_0 \cos 2\theta$ where $\theta = \tan^{-1} K_y/K_x$ is the angle between the total or CMM $\hbar\mathbf{K} = (\hbar K_x, \hbar K_y)$ of paired electrons in the CuO_2 plane and the a- (or x-) axis while Δ_0 is the value of the superconducting gap at the antinode ($\theta = 0, \pi/2$).[27,28] This behavior is also apparent in studies based on electronic Raman scattering[29] and in determinations of the in-plane magnetic penetration depth λ_{ab}.[30–32] Although a majority within the high-T_c community seems to argue for such non-s-wave pairing symmetry there are compelling dissenting views, particularly work within the past few years, by Müller,[33] Harshman *et al.*,[34] Klemm[35] and many others.

Here we apply a general l-wave BCS-type theory in a quasi-2D BEC picture with either $l = 0$ or $l = 2$ pairing symmetry. In Sec. 2 the l-wave BCS theory is discussed within the framework of the present model; in Sec. 3 we study a BEC of

linearly-dispersive, massless-like CPs in finite domains. We report explicit results for thermodynamic properties of a confined BEC, such as particle number density, energy density and specific heat. The density of SC charge carriers is estimated in Sec. 4 in terms of the magnetic penetration depth for linearly dispersive CPs. This leads to an analytic expression for the critical BEC temperature, which is then applied to various superconductors including YBCO with different doping levels. Discussion and conclusions are given in Sec. 5.

2. BCS Theory with l-Wave Pairs

An l-wave formulation of BCS theory was discussed long ago by Schrieffer[19] himself and studied in considerable detail by Anderson and Morel[36] in the weak-coupling limit. The d-wave extension in strong-coupling Eliashberg theory is reported in Refs. 37–39. The l-wave formulation of BCS theory has been successfully employed[27,28,40] to describe thermodynamic and transport properties of high-T_c cuprates (see also Refs. 19, 27, 28, 36, 40).

We briefly review Schrieffer's arguments showing that *independently of the coupling strength*, the presence of a Fermi sea induces a linear dispersion relation for CPs in an eigenstate of angular momentum l. A pair of fermions interacting via an isotropic potential V near the Fermi surface and with kinetic energies $\epsilon_k \equiv \hbar^2(k^2 - k_F^2)/2m^*$ (with $\hbar k_F$ the Fermi momentum) satisfy the Schrödinger equation in momentum space

$$(\mathcal{E}_{\mathbf{K}} - \epsilon_{\mathbf{k}+\mathbf{K}/2} - \epsilon_{\mathbf{k}-\mathbf{K}/2})a_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}'} \,, \tag{1}$$

where $\hbar K$ is the CMM, $\mathcal{E}_{\mathbf{K}}$ the energy eigenvalue (relative to the Fermi energy) and the interaction potential has matrix elements $V_{\mathbf{k}\mathbf{k}'} \equiv \langle \mathbf{k}, -\mathbf{k}|V|\mathbf{k}', -\mathbf{k}'\rangle$. For a given K, the relative-co-ordinate problem is isotropic and the interaction potential $V \equiv V(|\mathbf{r}|)$ admits a spherical harmonic expansion $V_{\mathbf{k}\mathbf{k}'} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} V_l(|\mathbf{k}|, |\mathbf{k}'|)Y_l^m(\Omega_{\mathbf{k}})Y_l^{-m}(\Omega_{\mathbf{k}'})$. By assuming that V_l as separable: $V_l(|\mathbf{k}|, |\mathbf{k}'|) = V_0^{(l)} f_k^l f_{k'}^{l*}$ so that the coupling interaction between different l-states is relatively weak, the problem admits an analytical solution. We introduce a Debye momentum $\hbar k_D$, a Debye energy $\hbar\omega_D \equiv \hbar^2 k_D^2/2m^*$ with m^* the electron effective mass and a Debye temperature $\Theta_D \equiv \hbar\omega_D/k_B$. Following similar steps as in the standard BCS theory one obtains a BCS-type integral relation for a CP in the eigenstate characterized by (l, m), namely:

$$\begin{aligned} V_0^{(l)} \sum_k \frac{1}{\mathcal{E}_K^{(l)} - \epsilon_{\mathbf{k}+\mathbf{K}/2} - \epsilon_{\mathbf{k}-\mathbf{K}/2}} \\ \simeq \mathcal{N}_0 V_0^{(l)} \int_{k_1}^{k_2} \frac{dk}{|\mathcal{E}_K^{(l)}| + \epsilon_{\mathbf{k}+\mathbf{K}/2} + \epsilon_{\mathbf{k}-\mathbf{K}/2}} = 1 \,, \end{aligned} \tag{2}$$

where the density of states per spin $\mathcal{N}_0 \approx$ const in the domain $k_F < |\mathbf{k} + \mathbf{K}/2|$, $|\mathbf{k} - \mathbf{K}/2| < \sqrt{k_F^2 + k_D^2}$ and $k_1 = k_F + (K/2)\cos\theta$, $k_2 = k_F + k_D - (K/2)\cos\theta$.

The integral in (2) yields the energy spectrum of excited CPs up to terms of order $(k_D/k_F)^2 = \Theta_D/T_F$[19]:

$$\mathcal{E}_K^{(l)} \simeq \mathcal{E}_0^{(l)} + c_1 \hbar K + O(K^2)\,, \tag{3}$$

where $\mathcal{E}_0^{(l)}$ is the binding energy of the CP ground state (with $\hbar K = 0$).[20]

$$\mathcal{E}_0^{(l)} = -2\hbar\omega_D/[\exp(2/\mathcal{N}_0 V_0^{(l)}) - 1]\,, \tag{4}$$

with $c_1 \equiv 2v_F/\pi$ in 2D, and $c_1 \equiv v_F/2$ in 3D. Clearly, the dispersion relation (3) is *linear* in leading order rather than quadratic as would be expected *in vacuo*. One direct consequence of this is that in order for a CP to remain bound, i.e., $\mathcal{E}_K^{(l)} < 0$, its maximum CM momentum $\hbar K$ must not exceed the pair-breaking threshold $\hbar K < \hbar K_0 \equiv |\mathcal{E}_0^{(l)}|/c_1$.[19]

The structure of the dispersion relation is such that a CP state of energy $\mathcal{E}_K^{(l)}$ is characterized only by a definite $\mathbf{K}$ but not definite $\mathbf{k}$. It follows that the total number of CPs in the system may be expressed in terms of the number operator

$$\hat{N} = \sum_{\mathbf{K}} \sum_{\mathbf{k}} \hat{B}_{\mathbf{k},\mathbf{K}^\dagger} \hat{B}_{\mathbf{k},\mathbf{K}} = \sum_{\mathbf{K}} \hat{n}_{\mathbf{K}}\,, \tag{5}$$

where $\hat{n}_{\mathbf{K}} = \sum_{\mathbf{k}} \hat{B}^\dagger_{\mathbf{k},\mathbf{K}} \hat{B}_{\mathbf{k},\mathbf{K}} \equiv \sum_{\mathbf{k}} \hat{n}_{\mathbf{k},\mathbf{K}}$ is the number operator for CPs with fixed CMM $\hbar\mathbf{K}$, while $\hat{B}_{\mathbf{k},\mathbf{K}}$, and $\hat{B}^\dagger_{\mathbf{k},\mathbf{K}}$ are CP annihilation and creation operators defined by $\hat{B}_{\mathbf{k},\mathbf{K}} \equiv \hat{c}_{\mathbf{k}+\mathbf{K}/2,\uparrow}\hat{c}_{-\mathbf{k}+\mathbf{K}/2,\downarrow}$ and $\hat{B}^\dagger_{\mathbf{k},\mathbf{K}} \equiv \hat{c}^\dagger_{\mathbf{k}+\mathbf{K}/2,\uparrow}\hat{c}^\dagger_{-\mathbf{k}+\mathbf{K}/2,\downarrow}$.[10,41] Here, $\hat{c}_{\mathbf{q},s}$, $\hat{c}^\dagger_{\mathbf{q},s}$ are Fermi operators of total momentum $\mathbf{q}$, and spin s, which satisfy the anti-commutation relations $\{\hat{c}_{\mathbf{k}+\mathbf{K}/2,s}, \hat{c}_{\mathbf{k}'+\mathbf{K}/2,s'}\} = 0 = \{\hat{c}^\dagger_{\mathbf{k}+\mathbf{K}/2,s}, \hat{c}^\dagger_{\mathbf{k}'+\mathbf{K}/2,s'}\}$, $\{\hat{c}_{\mathbf{k}+\mathbf{K}/2,s}, \hat{c}^\dagger_{\mathbf{k}'+\mathbf{K}/2,s'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}$. These anti-commutators allow showing that the operator $\hat{n}_{\mathbf{k},\mathbf{K}}$ is idempotent, i.e., $\hat{n}^2_{\mathbf{k},\mathbf{K}} = \hat{n}_{\mathbf{k},\mathbf{K}}$, with eigenvalues $n_{\mathbf{k},\mathbf{K}} = 0$ or 1. Consequently, the eigenvalues of $\hat{n}_{\mathbf{K}}$ are either $n_{\mathbf{K}} = 0, 1, 2, \ldots$ so that CPs may multiply occupy one and the same CMM state, including $\hbar K = 0$. We conclude that CPs obey Bose–Einstein statistics and therefore may suffer a BEC below a definite critical temperature T_c which can be identified with the superconducting temperature.

3. BEC in Constrained Domains

Accordingly, we evaluate the CP number and energy densities by introducing the exact mode density arising from the layered 2D structure of cuprates into the Bose–Einstein integrals

$$n = n_0(T) + \frac{1}{V}\int_{0^+}^{K_0} dK \frac{g(K)}{z^{-1}\exp\,\beta\mathcal{E}_K^{(l)} - 1}$$

$$\text{and} \quad u(T) = \frac{1}{V}\int_{0^+}^{K_0} dK \frac{g(K)\mathcal{E}_K^{(l)}}{z^{-1}\exp\,\beta\mathcal{E}_K^{(l)} - 1}\,. \tag{6}$$

Here $z \equiv \exp \beta\mu$ is the fugacity ($0 \leq z \leq 1$) with μ the chemical potential and $\beta \equiv 1/k_B T$, K_0 is the pair-breaking wavenumber and $\mathcal{E}_K^{(l)} = \hbar c_1 K$ ($K > 0$) is the dispersion relation for excited CPs. An important point is that superconducting pairing occurs mainly on the CuO_2 planes, so that the density of CM states is not simply given by the asymptotic Weyl expression $g(K) = VK^2/2\pi^2$. This expression is valid only in the large volume limit $V^{1/3}K \gg 1$, and thus neglects all finite-size effects on the thermodynamic properties of the system. Several thermodynamic properties of BECs in thin films were asymptotically evaluated by Pathria forty years ago[42] under a variety of boundary conditions, while a general study of spectra of finite systems was addressed by Baltes and Hilf.[43] Techniques based on the determination of the density of states of finite-size systems have been exploited by some of the authors of this paper to study the Casimir effect for quantum fields both at zero and finite temperature.[44–46] Previous results for BECs in constrained domains were also presented in Ref. 47.

The correct density of states can be determined by assuming that the CM component for an N-particle CP state is constrained within a rectangular domain with spatial dimensions a_1, a_2, a_3. We consider that these directions coincide with the a, b and c, crystallographic directions of a given cuprate. By assuming for definiteness that the CP CM-wavefunction satisfies periodic boundary conditions within that domain (Dirichlet or null-flux Neumann boundary conditions can be straightforwardly derived from the periodic solution), the CM density of states is $g(K) = \sum_{\{\mathbf{n}\}} \delta(K - K_{\mathbf{n}})$. Here, $K_{\mathbf{n}} = [(n_1\pi/a_1)^2 + (n_2\pi/a_2)^2 + (n_3\pi/a_3)^2]^{1/2}$. By using summation relations described in Refs. 44–46 the *exact* density of states may be rewritten in the convenient form:

$$g(K) = \frac{VK^2}{2\pi} \sum_{m_1,m_2,m_3} j_0(K u_{m_1,m_2,m_3}) \tag{7}$$

with $j_0(x)$ the zero-order spherical Bessel function and $u_{m_1,m_2,m_3} \equiv [(a_1 m_1)^2 + (a_2 m_2)^2 + (2a_3 m_3)^2]^{1/2}$. Note that Weyl's mode density is recovered in the large volume limit $a_1 a_2 a_3 \to \infty$. The integrals in (6) can be performed by changing to the variable $x \equiv \beta\hbar c_1 K$. For $x > 1$ they are rapidly convergent due to the combined action of the exponential factor of Bose distribution and a strongly oscillating mode density. In that case the upper integration limit may safely be taken as infinite and the integrals in (6) may be evaluated exactly. The number density (6) becomes:

$$n(T) = n_0(T) + \frac{(k_B T)^3}{\pi^2\hbar^2 c^3} {\sum_{m_1,m_2,m_3}}' \sum_{m=1}^{\infty} \frac{m z^m}{(m^2 + \alpha^2_{m_1,m_2,m_3})^2} \tag{8}$$

with $\alpha_{m_1,m_2,m_3} = (k_B T/\hbar c_1)[(m_1 a_1)^2 + (m_2 a_2)^2 + (m_3 a_3)^2]^{1/2}$, while the energy density is:

$$u(T) = \frac{(k_B T)^4}{\pi^2\hbar^2 c_1^3} {\sum_{m_1,m_2,m_3}}' \sum_{m=1}^{\infty} \frac{(3m^2 - \alpha^2_{m_1,m_2,m_3}) z^m}{(m^2 + \alpha^2_{m_1,m_2,m_3})^3} \,. \tag{9}$$

The primes on the summation signs in (8) and (9) mean that the term with $m_1 = m_2 = m_3 = 0$, corresponding to the asymptotic Weyl distribution, is omitted. By considering that CPs propagate within infinite extent layers in the a and b crystallographic directions, we keep only the terms $m_1 = m_2 = 0$ in (8) and (9). The remaining summation over m_3 may be performed analytically by introducing a reduced (or dimensionless) thickness variable $\eta \equiv k_B T\delta/\hbar c_1$ while δ is the thickness in length units, to give:

$$n(T) = n_0(T) + \frac{(k_B T)^3}{\pi^2\hbar^3 c_1^3}\Psi_3(z,\eta) \quad \text{and} \quad \mathrm{u(T)} = 3\frac{(\mathrm{k_B T})^4}{\pi^2\hbar^3 \mathrm{c}_1^3}\Phi_4(\mathrm{z},\eta)\,. \tag{10}$$

Here $\Psi_s(z,\eta) \equiv \sum_{m=1}^{\infty}(z^m/m^s)f_m(\eta)$, $\Phi_s(z,\eta) \equiv \sum_{m=1}^{\infty}(z^m/m^s)g_m(\eta)$, while $f_m(\eta) \equiv (1/2)[h_m(\eta) + (m\pi/\eta)\coth(m\pi/\eta)]$, $h_m(\eta) \equiv (m\pi/\eta)^2\sinh^{-2}(m\pi/\eta)$, and $g_m(\eta) \equiv (1/3)[(h_m(\eta) + (m\pi/\eta)\coth(m\pi/\eta)(1 + h_m(\eta))]$. Introducing the dimensionless variable $\eta_c \equiv k_B T_c\delta/\hbar c_1$ one finds that the critical temperature T_c follows from the conditions $n_0^B(T_c) \to 0$ and $z(T_c) \to 1$, with the result

$$k_B T_c = [\pi^2\hbar^3 c_1^3 n/\Psi_3(1,\eta_c)]^{1/3}\,. \tag{11}$$

The above expressions for the particle and energy densities yield well-known expressions for 3D and 2D condensates[10] by considering the limits $\eta \gg 1$, and $\eta \ll 1$, respectively. In particular, in the constant-volume molar heat capacity $C_V(T) = R(nk_B)^{-1}\partial u(T)/\partial T$ (with R the gas constant) involves a discontinuous jump in the 3D case $\Delta C_V = 6.57R$ at $T = T_c$, indicative of a second-order phase transition. On the other hand, in the 2D (thin-layer) limit we get $h_m(\eta) \to 0$, $f_m(\eta) \simeq m\pi/2\eta$, and $g_m(\eta) \simeq m\pi/3\eta$. Simple algebra leads to:

$$n^{2D}(T) = n_0^{2D}(T) + \frac{(k_B T)^2}{2\pi\hbar^2 c_1^2}\sum_{m=1}^{\infty}\frac{z^m}{m^2} \quad \text{and} \quad u^{2D}(T) = \frac{(k_B T)^3}{\pi\hbar^2 c_1^2}\sum_{m=1}^{\infty}\frac{z^m}{m^3}\,, \tag{12}$$

where the boson number density n^{2D} and energy per unit area u^{2D} are defined as $n^{2D} \equiv \delta n(T)$ and $u^{2D} \equiv \delta u(T)$. The critical BEC temperature is now given by:

$$k_B T_c^{2D} = \left[\frac{2\pi\hbar^2 c_1^2 n^{2D}}{\zeta(2)}\right]^{1/2}\,, \tag{13}$$

where $\zeta(2) = \pi^2/6$. In this case the molar heat capacity is $C_V(T) = [6R\zeta(3)/\zeta(2)](T/T_c)^2$ for $T < T_c$, but must be evaluated numerically for $T > T_c$. In contrast with the 3D case the heat capacity turns out to be continuous at $T = T_c$. It should be noted that in this case the ratio $C_V(T)/T \propto T$ for $T \leq T_c$. This *linear* behavior has been observed in a number of HTSC materials.[48] In fact, if we estimate the parameter η by introducing typical values for cuprates $k_B T_c \lesssim 100\ \mathrm{K} \simeq 10^{-2}$ eV, $\delta \simeq 3-4\times10^{-10}$ m, $c_1 \simeq 10^5\ \mathrm{ms}^{-1}$ ($\hbar \simeq 7\times10^{-16}$ eV–s), we get $\eta \lesssim 10^{-1}$, so that the quasi-2D limit may correctly approximate the thermodynamic properties of HTSC materials as in the present BCS-BEC model, especially in the regime of low T_cs (or small doping). Here we remain within this approximation.

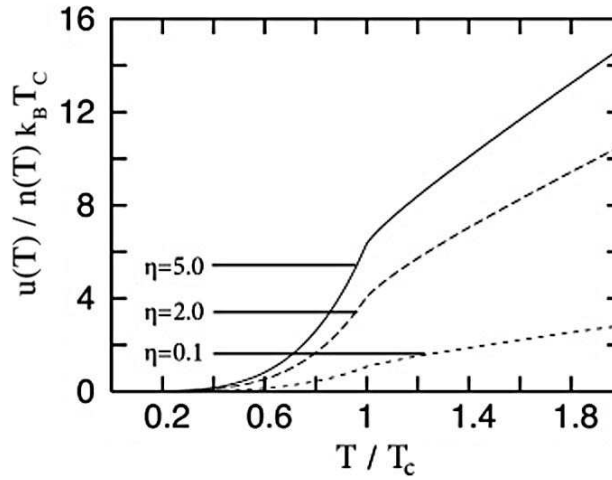

Fig. 1. Energy density $u(T)$ as function of T/T_c for several reduced layer widths $\eta \equiv k_B T\delta/\hbar c_1$.

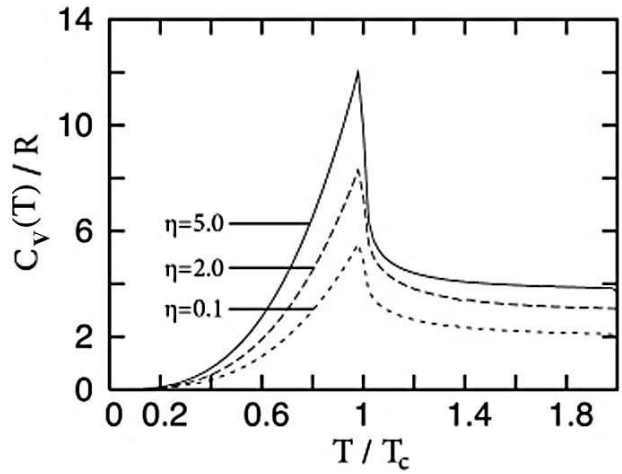

Fig. 2. Molar constant-volume heat capacity $C_V(T)$ as function of T/T_c for several reduced layer widths $\eta \equiv k_B T\delta/\hbar c_1$.

The relations just derived provide a complete description of the thermodynamic behavior of a constrained Bose gas system. Figure 1 is a plot of the internal energy as function of reduced temperature T/T_c for different values of the reduced layer width η. It reveals a marked variation with η. For $\eta \gg 1$ the internal energy develops an abrupt (but continuous) drop at T_c, while a gentler drop occurs in the opposite limit $\eta \ll 1$. This implies that the molar specific heat may radically alter its behavior depending on the layer width. In Fig. 2 we plot the specific heat for different values of the reduced layer width. We see that in the quasi-3D limit the

specific heat is discontinuous, leading to a second-order phase transition; in the quasi-2D limit it is continuous, implying a third-order phase transition. Then, the character of the BEC phase transition depends either on the underlying energy-momentum dispersion relation or on the effective dimensionality of the system.

4. Charge-Carrier Density and Critical Temperature

The density of SC charge carriers may be determined instead from knowledge of the magnetic penetration depth $\lambda_{ab}(T)$, i.e., the distance from the surface over which an external magnetic field decays within the superconductor along the crystallographic plane ab. This can be measured, for example, by MSR techniques.[6–8] A conventional Drude approach (not shown here) yields the explicit form of this parameter for linearly-dispersive CPs with charge $2e$, and moving with a uniform CM momentum P:

$$\frac{1}{\lambda_{ab}^2(T)} \equiv \frac{(2e)^2 4\pi c_1}{c^2 P}\, n(T)\,, \tag{14}$$

where e is the electron charge and c the speed of light (an equivalent expression has been derived by Fujita[10] by using phase invariance of the CP wavefunction). Since the momentum P may take any value within the interval $0 < P \leq \hbar K_0$ we fix it by introducing its mean value $P \to \bar{P} = \hbar K_0/2$. By additionally introducing in (14) the pair-breaking condition $K_0 = \mathcal{E}_0^{(l)}/\hbar c_1$ and the weak coupling BCS relation $|\mathcal{E}_0^{(l)}| = \Delta_0^{(l)2}/2\hbar\omega_D$ for arbitrary l symmetry,[36] we obtain:

$$n = \left[\frac{c^2}{64\pi c_1^2 e^2}\right]\left(\frac{\Delta_0^2}{\hbar\omega_D}\right)\frac{1}{\lambda_0^2}\,, \tag{15}$$

where $\lambda_0 \equiv \lambda_{ab}(0)$. Taking into account that the total number of bosonic CPs per unit volume n is made up mainly of those concentrated in narrow layers of width δ about the CuO_2 planes, it seems reasonable to consider $\delta = d$, where d is the interlayer spacing between adjacent CuO_2 planes (~ 3 Å in cuprates[50]). This assumption is consistent with contour plots of charge distributions derived from band-structure calculations for cuprates.[51,52] By introducing $n^{2D} = dn$ into expression (13) this finally leaves:

$$T_c = \frac{\hbar c}{4\pi k_B e}\left[\frac{3d}{\hbar\omega_D}\right]^{1/2}\frac{\Delta_0}{\lambda_0}\,, \tag{16}$$

which depends only on fundamental constants and physical variables accessible in principle by current experimental techniques. Interestingly, T_c depends linearly on Δ_0 and $1/\lambda_0$, in contrast with Uemura's empirical relation $T_c \propto 1/\lambda_0^2$. In fact, by introducing in (16) the YBCO parameters[50] $\Theta_D = 410$ K and $\Delta_0 = 17$ meV, as well as $d = 3.36$ Å, one gets the inverse linear relation $T_c = A/\lambda_0$, with $A = 14.24$ [$\mu\cdot$mK]. In Fig. 3 (adapted from Ref. 57) we compare the prediction given by this linear relation with measured values of T_c versus $1/\lambda_0$ for underdoped YBCO films, with dopings corresponding to T_cs ranging from 6 to 50 K (including also

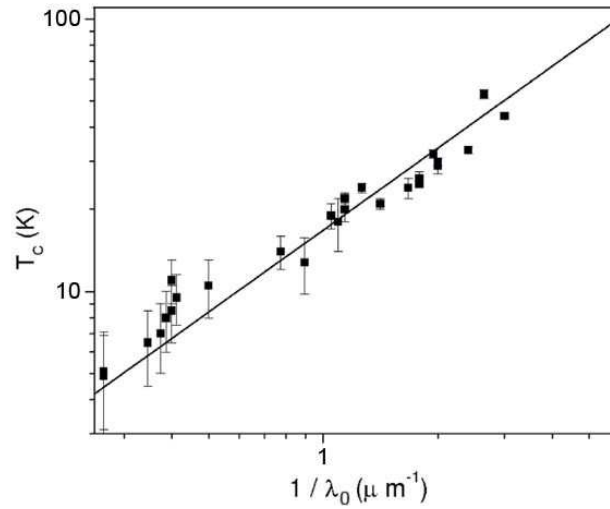

Fig. 3. Comparison of experimental T_cs versus theoretical predictions as function of zero-temperature inverse penetration length λ_0^{-1} for YBCO compounds with different dopings. Square datapoints are taken from Ref. 57, except for uppermost square referring to the optimally doped regime.[50] Vertical "error bars" represent full widths of σ_1 peaks where σ_1 is the real part of the conductivity σ employed in Ref. 57 to determine λ_0^{-1}.

one point corresponding to the optimally doped regime). This relation also gives an excellent description of measured data by Zuev *et al.*[57] which fall on the same curve defined by $T_c^{2.3\pm0.4} \propto \lambda_0^{-2}$, irrespective of annealing procedure, oxygen content, etc. The linear dependence of T_c on $1/\lambda_0$ had already been observed[58] in single YBCO crystals near the optimally-doped regime. Similarly, Broun *et al.*[32] report that their samples of high-purity single-crystal YBCO follow the rule $T_c \propto 1/\lambda_0 \propto (p-p_c)^{1/2}$, where the doping p is the number of holes per copper atom in the CuO_2 planes and p_c the minimal doping for superconductivity onset. The measured value of the penetration length in YBCO crystals is an order of magnitude larger than in thin films,[32,58] so that the specific values of T_cs derived from the linear relation are not in such good agreement as in the YBCO films. However, one should expect variations of parameters such as the energy gap associated to crystals and film systems. It has been pointed out[57] that YBCO *films* seem to behave more like other cuprate compounds such as BiSrCaCuO or LaSrCuO than do YBaCuO *single crystals*.

Theoretical values of T_c for superconducting cuprates with different compositions have been also calculated using (16). Here we report on the 11 layered-cuprate superconducting compounds $(La_{.925}Sr_{.075})_2CuO_4$, $YBa_2Cu_3O_{6.60}$, $YBa_2Cu_3O_{6.95}$, $YBa_2Cu_3O_{7-\delta}$, $(Y_{.95}Pr_{.05})Ba_2Cu_3O_{6.95}$, $(YPr_{40})Ba_2Cu_3O_{6.95}$, $YBa_2Cu_3O_8$, $Tl_2Ba_2Ca_2Cu_2O_8$, $Tl_2Ba_2Ca_2Cu_3O_{10}$, $Bi_2Sr_2Ca_2Cu_3O_{10}$ and $Bi_2Sr_2CaCu_2O_8$. Characteristic parameters for these materials were taken from tables compiled in Refs. 49, 50, 59–61. Table 1 shows results obtained using the foregoing assumptions, together with the physical parameters involved in the calculation. In most cases we

Table 1. Physical parameters of cuprate superconductors and predicted values for T_c, as well as the gap-to-T_c ratio $2\Delta_0/k_BT_c$. Here, $\Theta_D \equiv \hbar\omega_D/k_B$. Penetration depth λ_{ab} is estimated from plasma frequency via $\omega_p = c/\lambda_{ab}$. Parameters are taken from (a) Refs. 49, 50, 59 and references cited therein; (b) Ref. 50 Table 6.1; (c) Table 1 in Ref. 61; (d) supplementary information in Ref. 49; and (e) Ref. 59.

Superconductor	$T_c^{\text{exp (a)}}$	T_c^{th}	$\Theta_D^{\text{(b)}}$	$\Delta_0^{\text{(c)}}$	$\lambda_{ab}^{\text{(d)}}$	$d^{\text{(e)}}$	$\frac{2\Delta_0}{k_BT_c}^{\text{(exp)}}$	$\frac{2\Delta_0}{k_BT_c}^{\text{(th)}}$
$(La_{.925}Sr_{.075})_2CuO_4$	39	32.0	360	6.5	270	6.62	4.18	4.71
$YBa_2Cu_3O_{6.60}$	59	85.3	410	19.0	240	5.87	7.47	5.17
$YBa_2Cu_3O_{6.95}$	93.2	92.2	410	17.0	145	3.36	3.78	4.27
$YBa_2Cu_3O_7$	92	91.0	410	19.0	170	3.36	4.84	4.78
$(Y_{.95}Pr_{.05})Ba_2Cu_3O_7$	75	73.6	410	19.0	210	3.36	5.87	5.99
$(Y_{.60}Pr_{.40})Ba_2Cu_3O_7$	40	35.9	410	19.0	430	3.36	11.01	12.27
$YBa_2Cu_3O_8$	80	77.5	410	19.0	200	3.38	5.50	5.58
$Bi_2Sr_2CaCu_2O_8$	80	66.5	250	16.0	250	3.35	4.64	5.58
$Bi_2Sr_2Ca_2Cu_3O_{10}$	110	107.2	260	26.5	252	3.35	5.59	5.73
$Tl_2Ba_2Ca_2Cu_2O_8$	110	108.2	260	24	221	3.2	5.05	5.14
$Tl_2Ba_2Ca_2Cu_3O_{10}$	125	127.7	280	26	196	3.2	4.82	4.75

find rather satisfactory agreement between predicted and measured values of T_c. We also find very good agreement between theoretical and experimental gap-to-T_c ratios $2\Delta_0/k_BT_c$. We have not attempted to estimate uncertainties of our theoretical results since the accumulated data of some physical parameters appearing in (16), particularly Δ_0, show a wide scatter.

5. Discussion and Conclusions

To address cuprate high-T_c superconductors (HTSCs) we have presented a general BCS–BEC approach to study thermodynamic properties of boson particles constrained within finite quasi-2D spatial regions. A *linear*, as opposed to quadratic, dispersion relation is involved associated with the bosonic-CPs total or CMM. The continuous variation of thermodynamic variables when crossing over from the 2D to the 3D regime is explicitly shown. When merged with an l-wave BCS formalism for a quasi-2D BEC of moving CPs the approach describes features characteristic of HTSCs: (i) a simple formula for the critical transition temperature $T_c \propto 1/\lambda_{ab} \propto n^{1/2}$ (where λ_{ab} and n are the ab-plane penetration depth and charge-carrier number density, respectively) which applies to different cuprate SCs over a wide range of dopings and (ii) a molar constant-volume specific heat $C_V/T \propto T$, for $T \leq T_c$. Although the $T_c \propto 1/\lambda_{ab}$ behavior arising from our model apparently disagrees with the phenomenological Uemura relation $T_c \propto 1/\lambda_{ab}^2$,[6–8] different experimental studies[32,57,58] show a clear agreement with the inverse linear dependence of T_c on the SC density. It has been noted by Zuev *et al.*[57] that most of the data in the original Uemura's plot come from samples that are not as *severely* underdoped as the samples considered in their experimental studies. Conversely, Homes has remarked that Uemura's relation works reason-

ably well for underdoped materials, but not in the optimally doped or overdoped regime.[49]

Our results exhibit a weak dependence on the angular momentum state l, consistently with results derived by Maki *et al.*[27,28] for quantities averaged over a cylindrical Fermi surface. Furthermore, several authors[10,63] have proposed that the doping process could modify the electron-phonon interaction and the Fermi surface with a concomitant shift from d- to s-type coupling as doping increases. The strongest evidence for an s-wave order parameter in a cuprate is reviewed in Ref. 35 where several c-axis-twist experiments on BSCCO along with earlier c-axis tunneling between BSCCO/Pb junctions are surveyed. Reference 64 summarizes many of the problems with the so-called "phase-sensitive" tests[65] in YBCO. Additionally, predictions made in Ref. 66 that a vortex in a d-wave superconductor would exhibit a measurable density of states in a four-fold pattern emanating from the core have not been observed[67] in either YBCO or BSCCO. However, the vortex cores appear to be consistent[67] with isotropic s-waves.

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References

1. K. Tanaka *et al.*, *Science* **314**, 1914 (2006).
2. T. Valla *et al.*, *Science* **314**, 1910 (2006).
3. A. J. Millis, *Science* **314**, 1888 (2006).
4. H. B. Yang, *et al.*, *Nature* **456**, 77 (2008).
5. A. Kanigel *et al.*, *Phys. Rev. Lett.* **101**, 137002 (2008).
6. Y. J. Uemura *et al.*, *Phys. Rev. Lett.* **62**, 2317 (1989).
7. Y. J. Uemura *et al.*, *Phys. Rev. Lett.* **66**, 2665 (1991).
8. Y. J. Uemura, *J. Phys. Cond. Matt.* **16**, S4515 (2004).
9. Y. J. Uemura, *Physica B* **374–375**, 1 (2006).
10. S. Fujita and S. Godoy, *Theory of High Temperature Superconductivity* (Kluwer Academic Publishers, Dordrecht, 2002).
11. M. Casas *et al.*, *Phys. Lett. A* **245**, 55 (1998).
12. M. Casas *et al.*, *Physica C* **295**, 93 (1998).
13. S. Fujita, K. Ito and S. Godoy, *Quantum Theory of Conducting Matter, Superconductivity* (Springer-Verlag, Heidelberg, 2009).
14. M. de Llano and J. F. Annett, *Int. J. Mod. Phys. B* **21**, 3657 (2007).
15. T. A. Mamedov and M. de Llano, *J. Phys. Soc. Jpn.* **80**, 074718 (2011).
16. T. A. Mamedov and M. de Llano, *J. Phys. Soc. Jpn.* **79**, 044706 (2010).
17. J. Bardeen, L. N. Cooper and J. R. Schrieffer, *Phys. Rev.* **106**, 162 (1957).
18. J. Bardeen, L. N. Cooper and J. R. Schrieffer, *Phys. Rev.* **108**, 1175 (1957).

19. J. R. Schrieffer, *Theory of Superconductivity* (W.A. Benjamin, Reading, MA, 1964), p. 33.
20. L. N. Cooper, *Phys. Rev.* **104**, 1189 (1956).
21. P. C. Hohenberg, *Phys. Rev.* **158**, 383 (1967).
22. M. de Llano and J. J. Valencia, *Mod. Phys. Lett. B* **20**, 1067 (2006).
23. K. Miyake, *Prog. Theor. Phys.* **69**, 1794 (1983).
24. P. Gosdzinsky and R. Tarrach, *Am. J. Phys.* **59**, 70 (1991).
25. S. K. Adhikari *et al.*, *Phys. Rev. B* **62**, 8671 (2000).
26. M. Casas *et al.*, *Physica C* **295**, 93 (1998).
27. H. Won and K. Maki, *Phys. Rev. B* **49**, 1397 (1994).
28. K. Maki and H. Won, *J. Phys. I* (*France*) **6**, 2317 (1996).
29. M. Le Tacon *et al.*, *Nature Phys.* **2**, 537 (2006).
30. C. Panagopoulos, J. R. Cooper and T. Xiang, *Phys. Rev. B* **57**, 13422 (1998).
31. C. Panagapoulos and T. Xiang, *Phys. Rev. Lett.* **81**, 2336 (1998).
32. D. M. Broun *et al.*, *Phys. Rev. Lett.* **99**, 237003 (2007).
33. K. A. Müller, *Phil. Mag. Lett.* **82**, 279 (2002).
34. D. R. Harshman *et al.*, *Phys. Rev. B* **69**, 174505 (2004).
35. R. A. Klemm, *Phil. Mag.* **85**, 801 (2005).
36. P. W. Anderson and P. Morel, *Phys. Rev.* **123**, 1911 (1961).
37. E. Schachinger, J. J. Tu and J. P. Carbotte, *Phys. Rev. B* **67**, 214508 (2003).
38. E. Schachinger and J. P. Carbotte, *J. Phys. Studies* **7**, 209 (2003).
39. E. Schachinger and J. P. Carbotte, *Phys. Rev. B* **69**, 224501 (2004).
40. P. J. Hirschfeld and N. Goldenfeld, *Phys. Rev. B* **48**, 4219 (1993).
41. M. de Llano, F. J. Sevilla and S. Tapia, *Int. J. Mod. Phys. B* **20**, 2931 (2006).
42. R. K. Pathria, *Phys. Rev. A* **5**, 1451 (1972).
43. H. P. Baltes and E. R. Hilf, *Spectra of Finite Systems* (Bibliographisches Institut, Mannheim, 1976).
44. S. Hacyan, R. Jáuregui and C. Villarreal, *Phys. Rev. A* **47**, 4204 (1992).
45. C. Villarreal, *Phys. Rev. D* **51**, 2959 (1995).
46. R. Jáuregui, C. Villarreal and S. Hacyan, *Ann. Phys.* (*NY*) **321**, 2156 (2006).
47. C. Villarreal and M. de Llano, in *Proc. Ninth Conference on Quantum Field Theory under the Infuence of External Conditions* (World Scientific, Singapore, 2010) and arXiv,0912.1563v1.
48. R. A. Fisher, J. Gordon and N. E. Phillips, *J. Supercond.* **1**, 231 (1988).
49. C. C. Homes *et al.*, *Nature* **430**, 541 (2004).
50. C. P. Poole, H. A. Farach and R. J. Creswick, *Superconductivity* (Academic Press, London, 1995).
51. H. Krakauer, W. E. Pickett and R. E. Cohen, *J. Supercond.* **1**, 111 (1998).
52. W. E. Pickett, *Rev. Mod. Phys.* **61**, 433 (1989).
53. N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, (Holt, Rinehart and Winston, New York, 1976).
54. H. Haken, *Quantum Field Theory of Solids* (North-Holland, Amsterdam, 1976).
55. J. Orenstein *et al.*, *Phys. Rev. B* **42**, 6342 (1990).
56. J. F. Annett, *Superconductivity, Superfluids and Condensates* (Oxford University Press, New York, 2010).
57. Y. Zuev *et al.*, *Phys. Rev. Lett.* **95**, 137002 (2005).
58. T. Pereg-Barnea *et al.*, *Phys. Rev. B* **69**, 184513 (2004).
59. D. R. Harshman and P. Mills Jr., *Phys. Rev. B* **45**, 10684 (1992).
60. T. Hasegawa *et al.*, *Jpn. J. Appl. Phys.* **28**, L179 (1989).
61. D. M. Ginsberg (ed.), *Physical Properties of High Temperature Superconductors III* (World Scientific, Singapore, 1992).

62. R. Liang, D. A. Bonn and W. N. Hardy, *Phys. Rev. B* **93**, 180505 (2006).
63. E. A. Pashitiskii, V. I. Pentegov and A. V. Semenov, *Physica C* **282–287**, 1843 (1997).
64. G. B. Arnold and R. A. Klemm, *Phil. Mag.* **86**, 2811 (2006).
65. C. C. Tseui and J. R. Kirtley, *Revs. Mod. Phys.* **72**, 969 (2000).
66. M. Ichioka, A. Hasegawa and K. Machida, *Phys. Rev. B* **59**, 184 (1999).
67. R. A. Klemm, *Private Communication*.
68. T. Dahm *et al.*, *Nat. Phys.* **5**, 217 (2009).
69. C. Pfleiderer, *Rev. Mod. Phys.* **81**, 1551 (2009).