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ORIGINAL PAPER

Extended BCS-Bose Crossover

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Abstract Applying the generalized Bose-Einstein condensation (GBEC) formalism, we extend the BCS-Bose crossover theory by explicitly including hole Cooper pairs (2hCPs). From this, follows a phase diagram with two pure phases, one with 2hCPs and the other with electron Cooper pairs (2eCPs), plus a mixed phase with arbitrary proportions of 2eCPs and 2hCPs. The special-case phase when there is perfect symmetry, i.e., with ideal 50-50 proportions between 2eCPs and 2hCPs, corresponds to the usual BCS-Bose crossover. Explicitly including 2hCPs yields an *extended* BCS-Bose crossover which predicts improved T_c/T_F values for some conventional superconductors (i.e., with electron-phonon dynamics) when compared with experiment. To do this, we employ the BCS dimensionless coupling constant λ_{BCS} via the BCS gap equation and compare with the Bogoliubov et al. upper limit $\lambda_{BCS} \leq 1/2$. Another phase diagram presented exhibits experimental T_c/T_F values for some conventional superconductors for arbitrary proportions between 2eCPs and 2hCPs as function of $\Delta n = n/n_f - 1$, where n is the electron concentration and n_f that of unbound electrons at $T = 0$. The extended crossover is compared with experimental T_c/T_F values as well as to the gap-to-T_c ratio.

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1 Introduction

In the mid-60s, Keldysh et al. [1] assert that the weak Coulomb interaction corresponds to the assumption that the mean correlation energy q^2/r_D is much less than E_F, where q is the elemental electron charge and r_D the Debye screening radius, this condition being satisfied for the relatively small electron number-densities of $n \sim 10^{18}-10^{19}$ cm^{-3}. A year later Popov [2] suggested a theory of a Bose gas made up of bound pairs of Fermi particles which in the small density limit describes a system behaving as a Bose gas whose particles should form a Bose condensate at low-enough temperatures.

In 1967, Friedel et al. [3] proposed that "*two equations must be solved in the BCS formalism to obtain the gap equation at $T = 0$.*" A couple of years later Eagles [4] studied *two* simultaneous equations for the BCS gap Δ and its associated fermionic chemical potential μ. Solutions of these two simultaneous equations for T_c came to define a so-called *BCS-BEC crossover*. Leggett [5] later derived the *two* basic crossover equations at $T = 0$ [6] for *any* many-fermion system of identical particles each of mass m whose pair interaction is described by its S-wave scattering length a. At $T = 0$ he obtained the dimensionless *number equation*

$$\frac{4}{3} = \int_0^\infty d\widetilde{\epsilon}\sqrt{\widetilde{\epsilon}}\left[1 - \frac{\widetilde{\epsilon} - \widetilde{\mu}}{\sqrt{(\widetilde{\epsilon} - \widetilde{\mu})^2 + \widetilde{\Delta}^2}}\right] \tag{1}$$

(where tildes mean in units of the Fermi energy $E_F \equiv \hbar^2 k_F^2/2m$ of the associated ideal Fermi gas) and with μ and

Δ being the zero-temperature fermionic chemical potential and energy gap, respectively). He also obtained the dimensionless *gap equation* at $T = 0$

$$\frac{\pi}{k_F a} = \int_0^\infty d\widetilde{\epsilon}\left[\frac{1}{\sqrt{\widetilde{\epsilon}}} - \frac{\sqrt{\widetilde{\epsilon}}}{\sqrt{(\widetilde{\epsilon}-\widetilde{\mu})^2+\widetilde{\Delta}^2}}\right]. \tag{2}$$

These two equations were alternately derived as reported in Ref. [7]. Both these expressions are coupled transcendental equations to be solved self-consistently for μ and Δ with both quantities implicitly depending on a. Thus, the two equations are then valid for *any* coupling—weak, strong or intermediate. For weak coupling $\mu \simeq E_F$ as assumed by BCS [8]—whose theory is embodied in a *single* equation, the BCS gap equation. However, for very strong coupling, one must have $\mu \simeq -B_2/2$ with B_2 the two-body (positive) binding energy of a single pair *in vacuo*, all assuming that the two-body potential supports *one and only one* bound state as, e.g., the BCS model interaction can be shown [9] to effectively do so as well.

The BCS-Bose crossover was subsequently discussed by Nozières et al. [10], Ranninger et al. [11], Randeria et al. [12], van der Marel [13], Bar-Yam [14], Drechsler and Zwerger [15], Haussmann [16], Pistolesi and Strinati [17], among others.

We denote the crossover by "BCS-Bose" instead of by the more familiar "BCS-BEC" since a BEC cannot occur in either 2D nor in 1D [18] whereas bosons *can* form in both instances.

Boson-fermion (BF) models of SCs as a BEC go back to the mid-50s [19–22], pre-dating even the BCS-Bogoliubov theory [23–25]. BF models [19–22, 26–35] posit the existence of *actual* bosonic CPs. With a single exception [36] *all* BF models neglected the explicit effect of *hole* CPs included on an equal footing with electron CPs. This gave a *complete* BF model [36–39] at the heart of the GBEC theory to be described now.

2 GBEC Formalism

The GBEC formalism describes an ideal BF ternary gas consisting of unpaired electrons (fermions) with Cooper pairs of electrons (2eCP) and Cooper pairs of holes (2hCP) as bosons, *plus* very particular BF interactions. This formalism essentially given [36–39] by the Hamiltonian $H = H_0 + H_{int}$ where H_0 is the ideal ternary gas and H_{int} contains the BF vertex interactions [36, 37] pictured in Fig. 1.

Defining a simpler, reduced H_{int}, henceforth called H_{red}, by neglecting nonzero-K terms on the rhs renders an exactly diagonalizable expression. Then, ignoring these bosons with center-momentum-of-mass (CMM) $K \neq 0$ (in H_{int} but *not* also in H_0 as done in BCS theory) and using the Bogoliubov recipe of replacing b_0 and c_0 respectively by c-numbers $\sqrt{N_0}$ and $\sqrt{M_0}$ where N_0 and M_0 are the numbers of $K = 0$ bosons, one can readily exactly diagonalize [38] the reduced dynamical operator $\hat{H}_{red} - \mu\hat{N}$ via a Bogoliubov-Valatin transformation [40, 41] where $\hat{N}$ and μ are respectively the total-electron-number operator and chemical potential. Bringing the neglected CMM $K \neq 0$ terms back into the picture was recently accomplished via two-time Green function techniques [42–45].

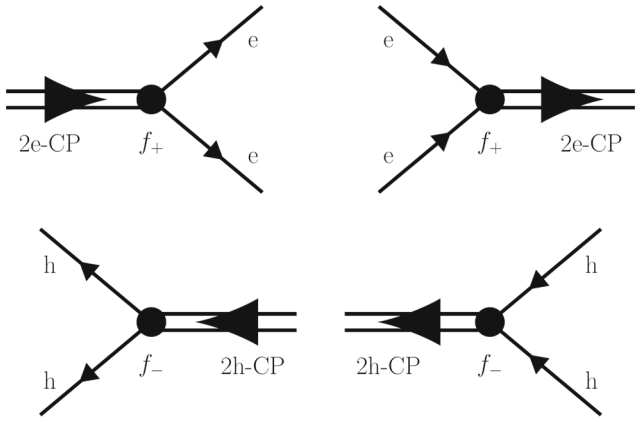

Fig. 1 The BF interaction hamiltonian from H_{int} [36, 37] consists of four interaction vertices each with two-fermion/one-boson creation-annihilation operators. They show how unpaired electrons (+) and/or holes (−) bind to form 2e or 2hCPs, or disintegrate into two unbound fermions

The thermodynamic (or Landau) potential in the grand canonical ensemble is then

$$\Omega(T, L^3, \mu, N_0, M_0) = -k_B T \ln\left[\mathrm{Tr}(\exp\{-\beta(\hat{H}_{red} - \mu\hat{N})\})\right] \tag{3}$$

where Tr is the "trace" of the diagonalized dynamical operator so that $\Omega(T, L^3, \mu, N_0, M_0)$ can now be evaluated explicitly. Here, T is the absolute temperature and $\beta \equiv 1/k_B T$. The Helmholtz free energy is then $F(T, L^3, \mu, N_0, M_0) \equiv \Omega(T, L^3, \mu, N_0, M_0) + \mu N$. Taking the negative partial derivative of (3) with respect to chemical potential, and also minimizing $F(T, L^3, \mu, N_0, M_0)$ wrt N_0 , M_0, gives

$$-\frac{\partial\Omega}{\partial\mu} = N \qquad \frac{\partial F}{\partial N_0} = 0 \qquad \frac{\partial F}{\partial M_0} = 0. \tag{4}$$

The first relation is familiar from statistical mechanics; here it ensures the net charge conservation of the GBEC formalism, i.e., *gauge invariance* [46]. This is in striking contrast

with BCS theory which lacks it. The last two requirements of (4) define a stable thermodynamic state.

The GBEC formalism via (4) leads to three coupled transcendental equations. These determine the thermodynamic-equilibrium phase diagram with three condensed phases: two pure BEC phases, one for 2eCPs the other for 2hCPs, and a mixed phase of arbitrary proportions. The three phases are determined numerically by solving the three coupled transcendental equations, and formally depend on three unknown functions: the electron chemical potential $\mu(T)$, along with the 2eCP and 2hCP Bose-Einstein (BE) condensate densities $n_0(T)$ and $m_0(T)$. From (4) one obtains two gap-like equations [36]

$$2\sqrt{n_0}[E_+(0)-2\mu]=\int_0^\infty d\epsilon N(\epsilon)\frac{\Delta(\epsilon)f_+(\epsilon)}{E(\epsilon)}\tanh\left[\tfrac{1}{2}\beta E(\epsilon)\right] \tag{5}$$

and

$$2\sqrt{m_0}[2\mu-E_-(0)]=\int_0^\infty d\epsilon N(\epsilon)\frac{\Delta(\epsilon)f_-(\epsilon)}{E(\epsilon)}\tanh\left[\tfrac{1}{2}\beta E(\epsilon)\right] \tag{6}$$

where $E_\pm(0)$ are the *phenomenological* energies of bosons with CMM $K = 0$, $E(\epsilon) \equiv \sqrt{(\epsilon-\mu)^2+\Delta^2(\epsilon)}$ is the familiar gapped Bogoliubov fermion dispersion relation with $\Delta(\epsilon) \equiv f_+\sqrt{n_0(T)} + f_-\sqrt{m_0(T)}$ where $N(\epsilon)$ is the electronic density of states and $f_\pm(\epsilon)$ are the BF vertex-function interactions as originally defined in Refs. [36, 37]. Additionally, the first of (4) yields the total number density

$$N/L^3 \equiv n = 2n_B(T) - 2m_B(T) + n_f(T) \tag{7}$$

where $n_f(T)$ is that of the *unpaired* electrons, while $n_B(T)$ and $m_B(T)$ are respectively those of 2e and 2hCPs in *all* bosonic states, ground together with excited, i.e., condensed and noncondensed. These turn out to be

$$n_B(T)\equiv n_0(T)+\int_{0+}^\infty d\varepsilon M(\varepsilon)\left[\exp\left[\beta(2E_f+\delta\epsilon-2\mu+\varepsilon)\right]-1\right]^{-1} \tag{8}$$

and

$$m_{B+}(T)\equiv m_0(T)+\int_{0+}^\infty d\varepsilon M(\varepsilon)\left[\exp\left[\beta(2\mu+\varepsilon-2E_f+\delta\epsilon)\right]-1\right]^{-1} \tag{9}$$

where $M(\varepsilon) \equiv (2m^{3/2}/\pi^2\hbar^3)\sqrt{\varepsilon}$ is the bosonic density of states. The Bose distributions here are clear reflections of the assumed bosonic nature of both kinds of CPs. Finally, the number density of *unpaired* electrons at any T turns out to be

$$\begin{aligned} n_f(T) \quad &\equiv \quad \int_0^\infty d\epsilon N(\epsilon)\left[1-\frac{\epsilon-\mu}{E(\epsilon)}\tanh\left\{\tfrac{1}{2}\beta E(\epsilon)\right\}\right] \\ &\xrightarrow[T\to 0]{} \left(2mE_f\right)^{3/2}/3\pi^2\hbar^3 \equiv n_f \end{aligned} \tag{10}$$

with the last result first reported in Ref. [47]. Here, E_f is a "pseudo-Fermi" energy; it coincides precisely with E_F only when $n_B(T) = m_B(T)$, i.e., ideal perfect 50-50 symmetry.

2.1 BCS and BEC Theories Subsumed in GBEC

The BCS and ordinary BEC theories are *subsumed* [36] as special cases in the GBEC formalism. If one assumes perfect 50-50 symmetry between 2e/2hCPs, i.e., $n_0(T) = m_0(T)$ *and* $n_{B+}(T) = m_{B+}(T)$, the (5) and (6) can be combined into a single equation. Then, setting $\mu \simeq E_F$ as assumed by BCS one readily arrives at the well-known BCS gap equation

$$1 = \frac{f^2N(\mu)}{2\delta\varepsilon}\int_0^{\delta\varepsilon} d\epsilon\frac{1}{\sqrt{\xi^2+\Delta(T)^2}}\tanh\left[\tfrac{1}{2}\beta\sqrt{\xi^2+\Delta(T)^2}\right] \tag{11}$$

where $\xi \equiv \epsilon - \mu$, provided one also identifies $\delta\varepsilon$ with $\hbar\omega_D$ and $f^2/2\delta\varepsilon$ with V, the BCS net attraction. In addition to this, one also recovers the $T = 0$ condensation energy exactly [48], and most recently as well [49] as for all $0 \le T \le T_c$.

For the *noninteracting* BF system, i.e., if $f_\pm(\epsilon) = 0$ in (5) and (6), one must put $\mu = E_f$, thus $E(\epsilon) = |\epsilon - E_f|$ with E_f again the pseudo-Fermi energy of the unpaired electrons. Then, the total electron number density for this noninteracting BF system is just

$$\begin{aligned} n \;=\; 2n_0(T) &+ \int_{0+}^\infty d\varepsilon M(\varepsilon)\left(\exp\beta[2E_f+\delta\varepsilon-2\mu+\varepsilon]-1\right)^{-1} \\ &+ \int_0^\infty d\epsilon N(\epsilon)\left[1-\frac{\epsilon-\mu}{E(\epsilon)}\left(1-2\exp[\beta E(\epsilon)]-1\right)^{-1}\right] \end{aligned} \tag{12}$$

where here one allows for the fully *a*symmetric case by ignoring the presence of 2hCPs altogether by setting

$m_B(T) = 0$ for all T. Determining $n_0(n, T)$ associated with the remaining 2eCP condensate fraction for $0 \leq T \leq T_c$, the critical temperature T_c is then defined as the temperature below which $n_0(n, T)$ just ceases to vanish as temperature decreases. The solution [36] is obtained from (12) since $n_0(n, T_c) = 0$, which yields

$$T_c \simeq 3.31 \frac{\hbar^2}{M k_B} \left(N_B/L^3\right)^{2/3}. \tag{13}$$

This is just the BEC formula for an ideal Bose gas with masses $M \equiv 2m$ and boson particle density $N_B/L^3 \equiv \frac{1}{2}(n - n_f)$ where $n_f \equiv n_f(T = 0)$ can be shown [47] to be the number density of unpaired electrons at $T = 0$. From this one arrives at the universal BEC ratio

$$T_c/T_F = (1/2)\,[2/3\Gamma(3/2)\zeta(3/2)]^{2/3} \simeq 0.218.$$

3 Extended BCS-Bose Crossover

As mentioned, in the 60's Friedel et al. [3] noted that BCS theory needs an additional equation whenever the chemical potential μ is allowed to differ from the Fermi energy E_F as must occur when coupling is *not* weak. They introduced a new expression to complete the picture, essentially a number equation for μ. Eagles [4] further clarified the picture by explicitly adding by hand a naturally obvious number equation to the BCS gap equation. This related the critical temperature T_c with carrier concentration n and became known as the BCS-BEC "*crossover*" theory which we instead call more correctly BCS-Bose.

Then, an *extended* BCS-Bose crossover emerges due to having postulated bosonic 2hCPs *explicitly* in addition to the 2eCPs. Hence, one now deals with the number densities of the condensed 2hCPs, $m_0(T)$, the excited 2hCPs, $m_{B+}(T)$, alongside the condensed 2eCPs $n_0(T)$ and the excited 2eCPs $n_{B+}(T)$. Having perfect symmetry between 2eCPs and 2hCPs, i.e., with half-and-half proportions, when $n_0(T) = m_0(T)$ *and* $n_{B+}(T) = m_{B+}(T)$ but for $\mu \neq E_F$ one recovers the original (still *unextended*) BCS-Bose crossover. Thus, one finds that the gap-like (5) can be summed with the gap-like (6) to yield a single equation. Explicitly, the equations of this crossover at $T = T_c$ implies that $\Delta(T_c) = 0$ so that $n_0(T_c) = m_0(T_c) = 0$ which leaves

$$[E_+(0) - E_-(0)] = f^2 \int_{E_f - \hbar\omega_D}^{E_f + \hbar\omega_D} d\epsilon N(\epsilon) \frac{1}{|\epsilon - \mu|} \tanh\left(\tfrac{1}{2}\beta_c|\epsilon - \mu|\right) \tag{14}$$

with $\beta_c \equiv 1/k_B T_c$, so the number equation is then

$$n = n_f(T_c) = \int_0^\infty d\epsilon N(\epsilon) \left[1 - \frac{\epsilon - \mu}{|\epsilon - \mu|}\left(1 - 2\,(\exp[\beta_c|\epsilon - \mu|] + 1)^{-1}\right)\right]. \tag{15}$$

Note that if $\mu = E_F$ identically one recovers the original BCS gap equation in the weak-coupling limit. Nevertheless, in this study, we solve the equations for *all* couplings including intermediate.

From Fig. 2, one notes that two coupling extremes are present. Weak coupling is around $n/n_f \simeq 1$ but in this regime one has very low T_c/T_Fs as implied already by BCS theory; this extreme results upon assuming $\mu = E_F$ with $n/n_f = 1$ (perfect ideal symmetry) and one then requires solving just one (gap-like) equation. On the other hand, strong coupling corresponds to $n/n_f \to \infty$, e.g., $n_f \to 0$ as in this extreme all electrons are paired, thus leaving a pure noninteracting Bose gas implying no interaction ($f = 0$) between unpaired electrons. This leads one to solve only the number equation.

Limits for large n/n_f are marked for each phase with a: (i) symbol $\diamond$ for the pure 2hGBEC phase with $T_c/T_F \to 1.507$, (ii) symbol $\circ$ for the familiar BCS-Bose

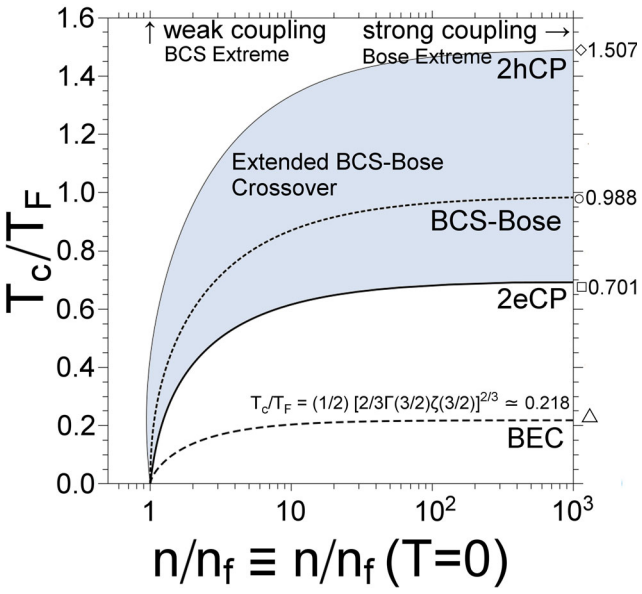

Fig. 2 T_c/T_F vs. n/n_f for all three condensed phases. The different phase boundaries associated with the extended crossover are evident. Thick curve, labeled 2eCPs phase, is obtained by simultaneously solving (5) with (7); thin curve labeled 2hCPs by solving (6) with (7) and short-dashed curve from simultaneously solving (5), (6) and (7) with perfect symmetry, i.e., the *unextended* BCS-Bose crossover. Long-dashed curve is BEC curve and is shown here for comparisons

crossover which corresponds to perfect ideal symmetry with $T_c/T_F \rightarrow 0.988$, and (iii) symbol □ for the pure 2eGBEC with $T_c/T_F \rightarrow 0.701$. In Fig. 2, the blue shaded area (online) bounded by pure 2h/2eGBEC phases is associated with the more general *mixed* extended crossover since one has arbitrary proportions of 2eCPs and 2hCPs. Inside this area the short-dashed curve associated with the special case of the extended crossover with *equal* proportions of 2eCPs and 2hCPs is shown; this coincides precisely with the usual *un*extended BCS-Bose crossover.

As T increases the entire weak-coupling system is driven to a crossover and finally to the strong-coupling regime; in this latter extreme region there remain no unpaired electrons leaving only a binary gas of bosonic 2eCPs and 2hCPs. As suggested by crossover authors, high-T_c superconductors can be expected to be in the crossover region where one has T_cs increased by several orders of magnitude higher than BCS theory predicts for conventional SCs. Resulting T_cs are at least three orders of magnitude higher than those of BCS, this dramatic T_c enhancement being due to the *mere presence of 2hCPs*. This behavior is analogous to the relativistic ideal Bose gas [50] where the mere presence of antibosons increases T_c wrt that with no antibosons present.

Then, there is an intermediate-coupling regime between weak- and strong-coupling. Furthermore, one sees that in the intermediate region T_c/T_F is enhanced wrt BCS theory, an aspect of the theory agreeing with Friedel et al. [3] that a crossover is needed to address A_3B compounds as well other kinds of SCs. An important role thus is being played by the dimensionless number density n/n_f because of the appreciable increase in T_cs solely by varying the charge-carrier density just as Eagles [4] suggested. In addition, the chemical potential exhibits the intermediate regime coupling for each phase of the extended crossover presented here and turns out to be very useful since one can then correctly *predict* T_c/T_Fs for some well-known elemental SCs.

Figure 3 shows experimental T_c/T_Fs (ninth column in Table 1) as function of $\Delta n \equiv n/n_f - 1$ and are compared with two pairs of theoretical curves of the extended crossover: a) top pair of curves labeled $\lambda_{BCS} = 1/2$ correspond to the Bogoliubov et al. upper limit with $\hbar\omega_D/E_F = 0.002$, b) bottom pair of curves are for $\lambda_{BCS} = 1/5$ with $\hbar\omega_D/E_F = 0.001$. These two values of $\hbar\omega_D/E_F$ are typical for conventional SCs; black dots refer to experimental values of T_c/T_F for each SC associated with perfect ideal symmetry between 2eCPs and 2hCPs, i.e., $\Delta n = 0$ (weak-coupling regime). One sees that SC empirical data of T_c/T_F falls within the theoretical curves of the extended crossover.

Table 1 lists some elemental SCs such as Nb, Hg, Al, In, Pb, and Sn. It shows the theoretical T_c/T_F predicted

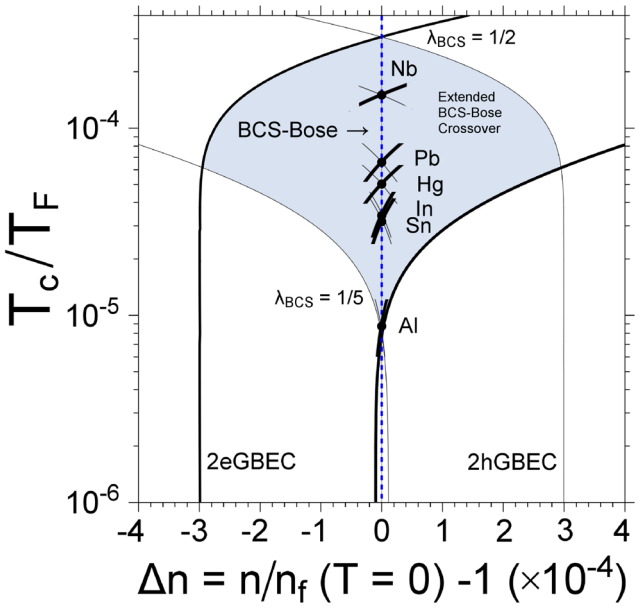

Fig. 3 Theoretical curves of extended crossover compared with experimental values of T_c/T_F for aforementioned SCs. Thick curves labeled 2eGBEC phase are obtained by simultaneously solving (5) with (7); thin curves labeled 2hGBEC by solving (6) with (7). Black dots mark experimental T_c/T_F values with $\Delta n = 0$ where error bars fall within dot size. Top pairs of curves labeled as $\lambda_{BCS} = 1/2$ (the Bogoliubov et al. upper limit, with $\hbar\omega_D/E_F = 0.002$) while bottom pairs of curves are for $\lambda_{BCS} = 1/5$ with $\hbar\omega_D/E_F = 0.001$. These Debye-energy values scaled with Fermi energy are typical for conventional superconductors

by the extended crossover for each SC and compared with experiment and with BCS theory. The BCS values of T_c/T_F are calculated via the BCS gap-T_c ratio $2\Delta(0)/k_BT_c \simeq 3.53$ using empirical data for the energy gap at $T = 0$. Extended crossover values for T_c/T_F are calculated by solving the three (5), (6) and (7) for perfect 50-50 symmetry, i.e., $n/n_f = 1$. Note that the extended crossover predicts critical temperatures for the aforementioned SCs quite well even for the so-called BCS theory "bad actors" [51]. Here, one solves three equations instead of the two equations suggested by Keldysh et al. [1], Popov [2], Friedel et al. [3], Eagles [4] and finally Leggett [5], rather than just one (the gap) equation with $\mu = E_F$ as implemented originally in BCS theory. Also shown is the gap-to-T_c ratio for the listed SCs (tenth column) this ratio being calculated from experimental data and is compared with that of the extended crossover (eleventh column). These values of the extended crossover are obtained by varying n/n_f slightly from unity (twelfth column) even for the BCS theory "bad actors" Hg and Pb. This follows the charge-carrier sign from Ref. [52] according to whether n/n_f is greater or less than 1.

Table 1 Experimental data for some conventional (i.e., presumed electron-phonon driven) SCs compared with results from the *extended* BCS-Bose crossover theory

	Θ_D	T_F	T_c	λ_{BCS}	$2\Delta(0)$	T_c/T_F ($\times 10^{-5}$)			$2\Delta(0)/k_BT_c$		
		($\times 10^5$)[a]				expt	BCS	Extended crossover	expt	Extended crossover	n/n_f
Al	394[a]	1.36	$(1.17 \pm 0.003)^e$	0.17	$(0.32 \pm 0.03)^j$	0.87	0.82	0.87	3.17	3.17	1.0000075
In	108[b]	1.00	$(3.41 \pm 0.001)^f$	0.28	$(1.05 \pm 0.03)^j$	3.40	3.64	3.42	3.57	3.57	1.0000480
Sn(w)	195[b]	1.18	$(3.72 \pm 0.001)^f$	0.24	$(1.11 \pm 0.03)^j$	3.15	3.26	3.13	3.46	3.46	1.0000410
Hg	88[b]	0.83	$(4.15 \pm 0.001)^d$	0.31	$(1.55 \pm 0.07)^i$	*5.00*	*6.48*	*4.99*	*4.33*	*4.33*	1.0000975
Pb	96[b]	1.10	$(7.20 \pm 0.8)^g$	0.37	$(2.68 \pm 0.06)^j$	*6.54*	*8.45*	*6.53*	*4.32*	*4.32*	1.0001263
Nb	276[b]	0.62	$(9.25 \pm 0.010)^c$	0.28	$(3.05 \pm 0.05)^h$	14.96	17.12	14.90	3.83	3.83	1.0002692

Debye (Θ_D), Fermi (T_F) and critical temperatures (T_c) are in kelvin units (K) and λ_{BCS} is the dimensionless BCS coupling parameter. Here, λ_{BCS} is determined via the BCS gap equation. Remarkably, the values obtained seem to corroborate the Bogoliubov et al. upper limit $\lambda_{BCS} \leq 1/2$. The BCS gap-to-T_c ratio formula $2\Delta(0)/k_BT_c \simeq 3.53$ was used to calculate BCS T_c/T_F values, using energy gap empirical data at $T = 0$ in meV units. T_c/T_F values predicted by the extended crossover are given for $n/n_f = 1$, while calculated $2\Delta(0)/k_BT_c$ values were adjusted with a n/n_f value near unity as shown in last column. Values in italics are the BCS "bad actors" [51]

Experimental data are from

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4 Conclusions

In the GBEC formalism, one defines a BCS-Bose crossover extended with explicit inclusion of 2hCPs. Starting from an ideal BF ternary gas with particular BF vertex interactions, the extended crossover is found to be defined by two thermodynamic-equilibrium requirements along with a well-known result from statistical mechanics that guarantees charge conservation. The extended crossover increases the critical temperature T_c several orders higher wrt BCS theory upon just slightly varying n/n_f around 1 and without abandoning electron-phonon dynamics.

Guided by the Bogoliubov et al. upper limit of $\lambda_{BCS} \leq 1/2$ one finds that the extended crossover predicts T_c/T_F values of some conventional SCs with perfect symmetry between 2eCPs and 2hCPs which agree reasonably well with experimental data. Also, the gap-to-T_c ratio is adjusted by varying the n/n_f very slightly around unity for the BCS "bad actors" Hg and Pb.

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